GRAM DETERMINANTS AND SEMISIMPLICITY CRITERIA FOR BIRMAN-WENZL ALGEBRAS

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Abstract. In this paper, we compute all Gram determinants associated to all cell modules of Birman-Wenzl algebras. As a by-product, we give a necessary and sufficient condition for Birman-Wenzl algebras being semisimple over an arbitrary field.

1. Introduction

In [3], Birman and Wenzl introduced a class of associative algebras $\mathscr{B}_n$, called Birman-Wenzl algebras, in order to study link invariants. They are quotient algebras of the group algebras of braid groups. On the other hand, there is a Schur-Weyl duality between $\mathscr{B}_n$ with some special parameters over $\mathbb{C}$ and quantum groups of types B, C, D [21]. Thus, $\mathscr{B}_n$ plays an important role in different disciplines.

In this paper, we work on $\mathscr{B}_n$ over the ground ring $R := \mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$ where $\omega = q - q^{-1}$ and q, r are indeterminates.

Definition 1.1. [3] The Birman-Wenzl algebra $\mathscr{B}_n$ is a unital associative R-algebra with generators $T_i, 1 \le i \le n-1$ and relations

a) $(T_i - q)(T_i + q^{-1})(T_i - r^{-1}) = 0$, for $1 \le i \le n-1$,
b) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $1 \le i \le n-2$,
c) $T_iT_j = T_jT_i$, for $|i-j| > 1$,
d) $E_iT_j^{\pm}E_i = r^{\pm}E_i$, for $1 \le i \le n-1$ and $j = i \pm 1$,
e) $E_iT_i = T_iE_i = r^{-1}E_i$, for $1 \le i \le n-1$,

where $E_i = 1 - \omega^{-1}(T_i - T_i^{-1})$ for $1 \le i \le n-1$.

In [15], Morton and Wassermann proved that $\mathscr{B}_n$ is isomorphic to Kauffman's tangle algebra [10] whose R-basis is indexed by Brauer diagrams. This enables them to show that $\mathscr{B}_n$ is a free R-module with rank $(2n-1)!!$. Let F be a field which contains non-zero elements $\mathbf{q}, \mathbf{r}$ and $\mathbf{q} - \mathbf{q}^{-1}$. Then the Birman-Wenzl algebra $\mathscr{B}_{n,F}$ over F is isomorphic to $\mathscr{B}_n \otimes_R F$. In this case, F is considered as an R-module such that r, q, ω act on F as $\mathbf{r}$, $\mathbf{q}$, and $\mathbf{q} - \mathbf{q}^{-1}$, respectively. We will use $\mathscr{B}_n$ instead of $\mathscr{B}_{n,F}$ if there is no confusion.

Let $\langle E_1 \rangle$ be the two-sided ideal of $\mathscr{B}_n$ generated by E_1. It is well-known that $\mathscr{B}_n / \langle E_1 \rangle$ is isomorphic to the Hecke algebra $\mathscr{H}_n$ associated to the symmetric group $\mathfrak{S}_n$. If we denote by $g_i, 1 \le i \le n-1$ the distinguished generators of $\mathscr{H}_n$, then the

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defining relations for $\mathscr{H}_n$ are as follows:

$$(g_i - q)(g_i + q^{-1}) = 0 \text{ for } 1 \le i \le n-1,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \text{ for } 1 \le i \le n-2,$$
$$g_i g_j = g_j g_i, \text{ for } |i-j| > 1.$$

The corresponding isomorphism from $\mathscr{B}_n/\langle E_1\rangle$ to $\mathscr{H}_n$ sends $T_i \pmod{\langle E_1\rangle}$ to g_i for all $1 \le i \le n-1$.

Using the cellular structure of $\mathscr{H}_n$ together with Morton-Wassermann's result in [15], Xi [22] proved that $\mathscr{B}_n$ is cellular over R in the sense of [8].

In [8], Graham and Lehrer constructed a class of generically irreducible modules for each cellular algebra, which are called cell modules. A question arises. When is a generically irreducible cell module not irreducible? Graham and Lehrer proved that any cell module of a cellular algebra is equal to its simple head if and only if the cellular algebra is (split) semisimple. This gives a method to determine the semisimplicity of a cellular algebra.

There is no result on the first problem for $\mathscr{B}_n$. In [21], Wenzl used the "Jones basic construction" and the Markov trace on $\mathscr{B}_n$ to give some partial results for $\mathscr{B}_n$ being semisimple. More explicitly, Wenzl [21, 5.6] proved that $\mathscr{B}_n$ is semisimple over $\mathbb{C}$ except possibly if q is a root of unity or $r = q^k$ for some $k \in \mathbb{Z}$. However, there is no explicit description for such k's.

Enyang constructed the Murphy basis for each cell module of $\mathscr{B}_n$ in [7] on which the Jucys-Murphy elements of $\mathscr{B}_n$ act upper triangularly. This enables us to use standard arguments (see e.g. [9] or more generally, [14]) to construct an orthogonal basis of $\mathscr{B}_n$. Via this orthogonal basis together with classical branching rule for $\mathscr{B}_n$ in [21], we obtain a recursive formula for the Gram determinant associated to each cell module of $\mathscr{B}_n$. This is the first main result of this paper.

Let $\Lambda^+(n)$ be the set of all partitions of n. When $r \notin \{q^{-1}, -q\}$, we will prove that $\mathscr{B}_n$ is semisimple if and only if

$$\prod_{k=2}^{n} \det G_{1,(k-2)} \det G_{1,(1^{k-2})} \prod_{\lambda \in \Lambda^+(n)} \det G_{0,\lambda} \neq 0.$$

Using our recursive formulae on Gram determinants, we compute $\det G_{1,\lambda}$ explicitly for $\lambda \in \cup_{k=2}^{n}\{(k-2), (1^{k-2})\}$. Note that $\prod_{\lambda\in\Lambda^+(n)} \det G_{0,\lambda} \neq 0$ if and only if $\mathscr{H}_n$ is semisimple. So, we can give a criterion for $\mathscr{B}_n$ being semsimple when $r \notin \{q^{-1}, -q\}$. When $r \in \{q^{-1}, -q\}$, we can determine whether $\mathscr{B}_n$ is semisimple by elementary computation. It gives a complete solution of the problem on the semisimplicity of $\mathscr{B}_n$ over an arbitrary field. This is the second main result of the paper.

Note that the group algebra of $\mathfrak{S}_n$ is both a subalgebra and a quotient algebra of the Brauer algebra B_n [4]. Thus, Doran-Wales-Hanlon[6] can restrict a module for B_n to the group algebra of $\mathfrak{S}_n$. However, $\mathscr{H}_n$ is not a subalgebra of $\mathscr{B}_n$. We can not restrict a $\mathscr{B}_n$-module to $\mathscr{H}_n$. In other words, we can not use the method in [16, 17] to give a criterion for $\mathscr{B}_n$ being semisimple. Finally, we remark that the method

we use in the current paper can be used to deal with cyclotomic Nazarov-Wenzl algebras [1]. Details will appear elsewhere.

We organize this paper as follows. In section 2, we recall the Jucys–Murphy basis for each cell module of $\mathscr{B}_n$ in [7]. An orthogonal basis of each cell module of $\mathscr{B}_n$ will be constructed in section 3. In section 4, we prove the recursive formulae on Gram determinants. Finally, we give a criterion for $\mathscr{B}_n$ being semisimple in section 5.

2. Jucys-Murphy basis for $\mathscr{B}_n$

In this section, unless otherwise stated, we assume $R = \mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$ where $\omega = q - q^{-1}$ and q, r are indeterminates. The main purpose of this section is to construct the Jucys-Murphy basis of $\mathscr{B}_n$ by using Enyang's basis of each cell module of $\mathscr{B}_n$. We state some identities needed later on. We start by recalling the definition of Jucys-Murphy elements $L_i, 1 \le i \le n$ for $\mathscr{B}_n$ in [7].

Define $L_1 = r$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$ for $2 \le i \le n$[1]. The following identities can be found in [3] and [7].

Lemma 2.1 ([3, 7])**.** *Suppose* $\delta = \frac{(q+r)(qr-1)}{r(q+1)(q-1)}$. *We have:*

a) $E_i^2 = \delta E_i$, $1 \le i \le n-1$,
b) $E_iT_j = T_jE_i$, $|i-j| > 1$,
c) $T_i^2 = 1 + \omega(T_i - r^{-1}E_i)$, $1 \le i \le n-1$,
d) $E_iE_jE_i = E_i$ *for* $1 \le i \le n-1$ *and* $j = i \pm 1$,
e) $E_iE_j = T_jT_iE_j = E_iT_jT_i$ *for* $1 \le i \le n-1$ *and* $j = i \pm 1$,
f) $T_iL_k = L_kT_i$ *if* $k \notin \{i, i+1\}$,
g) $E_iL_k = L_kE_i$ *if* $k \notin \{i, i+1\}$,
h) $L_iL_k = L_kL_i$ *for all* $1 \le i, k \le n$,
i) $T_iL_iL_{i+1} = L_iL_{i+1}T_i$ *for all* $1 \le i \le n-1$,
j) $L_2L_3 \cdots L_n$ *is a central element in* $\mathscr{B}_n$.

The following result is well-known. One can prove it by checking the defining relations for $\mathscr{B}_n$ in Definition 1.1.

Lemma 2.2. *a) There is a quasi R-linear automorphism* $\sigma : \mathscr{B}_n \to \mathscr{B}_n$ *such that* $\sigma(T_i) = T_i^{-1}$, $\sigma(q) = q^{-1}$, $\sigma(r) = r^{-1}$. *Therefore,* $\sigma(\delta) = \delta$, $\sigma(E_i) = E_i$ *and* $\sigma(L_j) = L_j^{-1}$ *for* $1 \le i \le n-1$ *and* $1 \le j \le n$.
b) There is an R-linear anti-involution $* : \mathscr{B}_n \to \mathscr{B}_n$ *such that* $T_i^* = T_i$. *Thus,* $E_i^* = E_i$ *and* $L_j^* = L_j$ *for* $1 \le i \le n-1$ *and* $1 \le j \le n$.

Lemma 2.3. *Suppose that k is a positive integer. The following equalities hold.*

a) $L_iL_{i+1}E_i = E_i = E_iL_iL_{i+1}$, $1 \le i \le n-1$.
b) $T_{i-1}L_i^k = L_{i-1}^kT_{i-1} + \omega\sum_{j=1}^k L_{i-1}^{j-1}(1 - E_{i-1})L_i^{k-j+1}$, $2 \le i \le n$.
c) $T_iL_i^k = L_{i+1}^kT_i - \omega\sum_{j=1}^k L_{i+1}^{j}(1 - E_i)L_i^{k-j}$, $1 \le i \le n-1$.
d) $T_{i-1}L_i^{-k} = L_{i-1}^{-k}T_{i-1} - \omega\sum_{j=1}^k L_{i-1}^{-j}(1 - E_{i-1})L_i^{j-k}$, $2 \le i \le n$.

[1] In [7], Enyang defined $L_1 = 1$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$.

e) $T_iL_i^{-k} = L_{i+1}^{-k}T_i + \omega\sum_{j=1}^{k} L_{i+1}^{1-j}(1-E_i)L_i^{j-k-1}$, $1 \le i \le n-1$.

Proof. (a) can be proved by induction on i. Note that $T_{i-1}L_i = T_{i-1}^2 L_{i-1}T_{i-1}$. Now, (b) follows from Definition 1.1(e) and Lemma 2.1(c) for $k=1$. In general, by induction assumption,

$$T_{i-1}L_i^k = T_{i-1}L_i^{k-1}L_i = (L_{i-1}^{k-1}T_{i-1} + \omega\sum_{j=1}^{k-1} L_{i-1}^{j-1}(1-E_{i-1})L_i^{k-j})L_i. \tag{2.4}$$

So, (b) follows if we use (b) for $k=1$ to rewrite $T_{i-1}L_i$ in (2.4). One can verify (c) similarly. Applying σ to (b) (resp. (c)) and using Lemma 2.1(c) yields (d) (resp. (e)). □

Lemma 2.5. *For any $1 \le i \le n-1$, and $k \in \mathbb{N}$,*

$$E_iL_i^kE_i = r^2E_iL_i^{-k}E_i + r\omega\sum_{j=1}^{k-1}(E_iL_i^{-j}E_iL_i^{k-j}E_i - E_iL_i^{k-2j}E_i)$$

Proof. By induction on i, we have

$$E_iL_iE_i = r(\delta + \omega\sum_{j=1}^{i-1}(L_j - L_j^{-1}))E_i. \tag{2.6}$$

Applying σ to $E_iL_iE_i$ and using (2.6) yields $r^2E_iL_i^{-1}E_i = E_iL_iE_i$. This proves the result for $k=1$. In general, we have $E_iL_i^kE_i = rE_iT_iL_i^kE_i$. Using Lemma 2.3(a)(c) and Definition 1.1 to simplify $E_iT_iL_i^kE_i$ yields the formula as required. □

For any R-algebra A, let $Z(A)$ be the center of A.

Proposition 2.7. *Given a positive integer $i \le n-1$ and an integer k. We have $E_iL_i^kE_i = \omega_i^{(k)}E_i$, where $\omega_i^{(k)} \in R[L_1^{\pm}, L_2^{\pm}, \cdots, L_{i-1}^{\pm}] \cap Z(\mathscr{B}_{i-1})$.*

Proof. By Lemma 2.5, we can assume that $k \ge 0$ without loss of generality. The case $k=0$ is trivial since $E_i^2 = \delta E_i$. We prove the result by induction on i and k for $k>0$. Since we are assuming that $L_1 = r$, $\omega_1^{(k)} = r^k\delta$. When $k=1$, the result follows from (2.6). Now, we assume that $i>1$ and $k>1$.

Write $L_i^k = T_{i-1}L_{i-1}T_{i-1}L_i^{k-1}$. By Lemma 2.3(b),

$$E_iL_i^kE_i = E_iT_{i-1}L_{i-1}^kT_{i-1}E_i + \omega\sum_{j=1}^{k-1}E_iT_{i-1}L_{i-1}^j(1-E_{i-1})L_i^{k-j}E_i \tag{2.8}$$

First, we consider $\sum_{j=1}^{k-1}E_iT_{i-1}L_{i-1}^j(1-E_{i-1})L_i^{k-j}E_i$. Applying $*$ to Lemma 2.3(c) yields $L_{i-1}^kT_{i-1} = T_{i-1}L_i^k - \omega\sum_{j=1}^{k}L_{i-1}^{k-j}(1-E_{i-1})L_i^j$. Multiplying E_iE_{i-1} (resp. E_i) on the left (resp. right) of $L_{i-1}^kT_{i-1}$ and using Definition 1.1(e), Lemma 2.3(a) and Lemma 2.1(d) together with induction assumption on $E_{i-1}L_{i-1}^jE_{i-1}$ for $j<k$, we have

$$E_iE_{i-1}L_{i-1}^kT_{i-1}E_i = r^{-1}L_{i-1}^{-k}E_i - \omega\sum_{j=1}^{k}(L_{i-1}^{k-2j} - \omega_{i-1}^{(k-j)}L_{i-1}^{-j})E_i \tag{2.9}$$

Similarly, we have

$$E_iT_{i-1}L_i^kE_i = rL_{i-1}^kE_i + \omega\sum_{j=1}^{k}(L_{i-1}^{j-1}\omega_i^{(k-j+1)} - L_{i-1}^{2j-2-k})E_i \tag{2.10}$$

Applying $*$ on both sides of (2.9) and using (2.10), we have

$$\sum_{j=1}^{k-1} E_iT_{i-1}L_{i-1}^j(1-E_{i-1})L_i^{k-j}E_i = E_if_1 \tag{2.11}$$

for some $f_1 \in R[L_1^{\pm}, L_2^{\pm}, \cdots, L_{i-1}^{\pm}]$. Now, we discuss $E_iT_{i-1}L_{i-1}^kT_{i-1}E_i$. We have

$$\begin{aligned} E_iT_{i-1}L_{i-1}^kT_{i-1}E_i &= E_iE_{i-1}T_i^{-1}L_{i-1}^kT_{i-1}E_i \\ &= E_iE_{i-1}(T_i - \omega(1-E_i))L_{i-1}^kT_{i-1}E_i \end{aligned}$$

Note that $E_iE_{i-1}T_iL_{i-1}^kT_{i-1}E_i = E_iE_{i-1}L_{i-1}^kE_{i-1}E_i = \omega_{i-1}^{(k)}E_i$ and $E_iE_{i-1}E_iL_{i-1}^kT_{i-1}E_i = rL_{i-1}^kE_i$. By (2.9), together with (2.11), we have $E_iL_i^kE_i = \omega_i^{(k)}E_i$, where $\omega_i^{(k)} \in R[L_1^{\pm}, L_2^{\pm}, \cdots, L_{i-1}^{\pm}]$. We close the proof by showing that $\omega_i^{(k)} \in Z(\mathscr{B}_{i-1})$. Note that any element $h \in \mathscr{B}_{i-1}$ commutes with E_i and L_i. We have $hE_iL_i^kE_i = E_iL_i^kE_ih$, which implies $E_ih\omega_i^{(k)} = E_i\omega_i^{(k)}h$. If we identify the monomials of $\mathscr{B}_{i+1}$ with Kauffman's tangles, we have $hE_i = 0$ for $h \in \mathscr{B}_{i-1}$ if and only if $h = 0$. Thus $h\omega_i^{(k)} = \omega_i^{(k)}h$ for all $h \in \mathscr{B}_{i-1}$. □

In the remainder of this section, we are going to construct the Jucys-Murphy basis of $\mathscr{B}_n$. We start by recalling some combinatorics.

Recall that a partition of n is a weakly decreasing sequence of non–negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$. In this case, we write $\lambda \vdash n$. The set $\Lambda^+(n)$, which consists of all partitions of n, is a poset with dominance order $\trianglelefteq$ as the partial order on it. Given $\lambda, \mu \in \Lambda^+(n)$, $\lambda \trianglelefteq \mu$ if $\sum_{j=1}^i \lambda_j \le \sum_{j=1}^i \mu_j$ for all possible i. Write $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.

Suppose that λ and μ are two partitions. We say that μ is obtained from λ by adding a box if there exists an i such that $\mu_i = \lambda_i + 1$ and $\mu_j = \lambda_j$ for $j \neq i$. In this situation we will also say that λ is obtained from μ by removing a box and we write $\lambda \to \mu$ and $\mu \setminus \lambda = (i, \lambda_i + 1)$. We will also say that the pair $(i, \lambda_i + 1)$ is an addable node of λ and a removable node of μ. Note that $|\mu| = |\lambda| + 1$.

The Young diagram $Y(\lambda)$ for a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a collection of boxes arranged in left-justified rows with λ_i boxes in the i-th row of $Y(\lambda)$. A λ-tableau $\mathbf{s}$ is obtained by inserting $i, 1 \le i \le n$ into $Y(\lambda)$ without repetition. The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{s}$ by permuting its entries. Let $\mathbf{t}^\lambda$ be the λ-tableau obtained from the Young diagram $Y(\lambda)$ by adding $1, 2, \cdots, n$ from left to right along each row and from top to bottom along each column. If $\mathbf{t}^\lambda w = \mathbf{s}$, write $w = d(\mathbf{s})$. Note that $d(\mathbf{s})$ is uniquely determined by $\mathbf{s}$.

A λ-tableau $\mathbf{s}$ is standard if the entries in $\mathbf{s}$ are increasing both from left to right in each row and from top to the bottom in each column. Let $\mathscr{T}_n^{std}(\lambda)$ be the set of all standard λ-tableaux.

Given an $\mathbf{s} \in \mathscr{T}_n^{std}(\lambda)$, let $\mathbf{s}\downarrow_i$ be obtained from $\mathbf{s}$ by removing all the entries j in $\mathbf{s}$ with $j > i$. Let $\mathfrak{s}_i$ be the partition of i such that $\mathbf{s}\downarrow_i$ is an $\mathfrak{s}_i$-tableau. Then

$\mathfrak{s} = (\mathfrak{s}_0, \mathfrak{s}_1, \cdots, \mathfrak{s}_n)$ is a sequence of partitions such that $\mathfrak{s}_i \to \mathfrak{s}_{i+1}$. Conversely, if we insert i into the box $\mathfrak{s}_i \setminus \mathfrak{s}_{i-1}$, then we obtain an $\mathbf{s} \in \mathscr{T}_n^{std}(\lambda)$. Thus, there is a bijection between $\mathscr{T}_n^{std}(\lambda)$ and the set of all $(\mathfrak{s}_0, \mathfrak{s}_1, \cdots, \mathfrak{s}_n)$ such that $\mathfrak{s}_i \to \mathfrak{s}_{i+1}, 0 \le i \le n-1$ and $\mathfrak{s}_0 = 0$, and $\mathfrak{s}_n = \lambda$.

Recall that $\mathfrak{S}_n$ is generated by $s_i, 1 \le i \le n-1$ subject to the relations (1) $s_i^2 = 1$, $1 \le i \le n-1$ (2) $s_i s_j = s_j s_i$ if $|i-j| > 1$ (3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \le i \le n-2$. Assume that $0 \le f \le \lfloor \frac{n}{2} \rfloor$. Let $\mathfrak{S}_{n-2f}$ be the subgroup of $\mathfrak{S}_n$ generated by s_j, $2f+1 \le j \le n-1$. Following [7], let $\mathfrak{B}_f$ be the subgroup of $\mathfrak{S}_n$ generated by $\tilde{s}_i, \tilde{s}_0$, where $\tilde{s}_i = s_{2i} s_{2i-1} s_{2i+1} s_{2i}$, $1 \le i \le f-1$ and $\tilde{s}_0 = s_1$. Enyang [7] proved that $\mathcal{D}_{f,n}$ is a complete set of right coset representatives of $\mathfrak{B}_f \times \mathfrak{S}_{n-2f}$ in $\mathfrak{S}_n$, where

$$\mathcal{D}_{f,n} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} (2i+1)w < (2j+1)w, \, (2i+1)w < (2i+2)w, \\ 0 \le i < j < f, \text{ and } (k)w < (k+1)w, \, 2f < k < n \end{array} \right\}.$$

For $\lambda \vdash n-2f$, let $\mathfrak{S}_\lambda$ be the Young subgroup of $\mathfrak{S}_{n-2f}$ generated by s_j, $2f+1 \le j \le n-1$ and $j \ne 2f + \sum_{k=1}^{i} \lambda_k$ for all possible i. A standard λ-tableau $\hat{\mathbf{s}}$ is obtained by using $2f+i, 1 \le i \le n-2f$ instead of i in the usual standard λ-tableau $\mathbf{s}$. Define $d(\hat{\mathbf{s}}) \in \mathfrak{S}_{n-2f}$ by declaring that $\hat{\mathbf{s}} = \hat{\mathbf{t}}^\lambda d(\hat{\mathbf{s}})$. By abuse of notation, we denote by $\mathscr{T}_n^{std}(\lambda)$ the set of all standard λ-tableaux $\hat{\mathbf{s}}$.

It has been proved in [22] that $\mathscr{B}_n$ is a cellular algebra over a commutative ring. In what follows, we recall Enyang's cellular basis for $\mathscr{B}_n$.

Let $\Lambda_n = \{ (f, \lambda) \mid \lambda \vdash n-2f, 0 \le f \le \lfloor \frac{n}{2} \rfloor \}$. Given $(k, \lambda), (f, \mu) \in \Lambda_n$, define $(k, \lambda) \trianglelefteq (f, \mu)$ if either $k < f$ or $k = f$ and $\lambda \trianglelefteq \mu$. Write $(k, \lambda) \triangleleft (f, \mu)$, if $(k, \lambda) \trianglelefteq (f, \mu)$ and $(k, \lambda) \ne (f, \mu)$.

For any $w \in \mathfrak{S}_n$, write $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$ if $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w. It is well-known that T_w is independent of a reduced expression of w. Let $I(f, \lambda) = \mathscr{T}_n^{std}(\lambda) \times \mathcal{D}_{f,n}$ and define

$$C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} = T_u^* T_{d(\mathbf{s})}^* \mathfrak{M}_\lambda T_{d(\mathbf{t})} T_v, \quad (\mathbf{s}, u), (\mathbf{t}, v) \in I(f, \lambda) \tag{2.12}$$

where $\mathfrak{M}_\lambda = E^f X_\lambda$, $E^f = E_1 E_3 \cdots E_{2f-1}$, $X_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} T_w$, and $l(w)$, the length of $w \in \mathfrak{S}_n$.

Theorem 2.13. [7] *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over R. Let $* : \mathscr{B}_n \to \mathscr{B}_n$ be the R-linear anti-involution in Lemma 2.2. Then*

a) *$\mathscr{C}_n = \left\{ C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} \mid (\mathbf{s}, u), (\mathbf{t}, v) \in I(f, \lambda), \lambda \vdash n-2f, 0 \le f \le \lfloor \frac{n}{2} \rfloor \right\}$ is a free R–basis of $\mathscr{B}_n$.*

b) *$*(C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)}) = C^{(f,\lambda)}_{(\mathbf{t},v)(\mathbf{s},u)}$.*

c) *For any $h \in \mathscr{B}_n$,*

$$C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} h \equiv \sum_{(\mathbf{u},w) \in I(f,\lambda)} a_{\mathbf{u},w} C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{u},w)} \mod \mathscr{B}_n^{\triangleright(f,\lambda)}$$

where $\mathscr{B}_n^{\triangleright(f,\lambda)}$ is the free R-submodule generated by $C^{(k,\mu)}_{(\tilde{\mathbf{s}},\tilde{u})(\tilde{\mathbf{t}},\tilde{v})}$ with $(k, \mu) \triangleright (f, \lambda)$ and $(\tilde{\mathbf{s}}, \tilde{u}), (\tilde{\mathbf{t}}, \tilde{v}) \in I(k, \mu)$. Moreover, each coefficient $a_{\mathbf{u},w}$ is independent of $(\mathbf{s}, u)$.

Theorem 2.13 shows that $\mathscr{C}_n$ is a cellular basis of $\mathscr{B}_n$ in the sense of [8]. In this paper, we will only consider right modules.

By general theory about cellular algebras in [8], we know that, for each $(f,\lambda) \in \Lambda_n$, there is a cell module $\Delta(f,\lambda)$ of $\mathscr{B}_n$, spanned by

$$\{\, \mathfrak{M}_\lambda T_{d(\mathfrak{t})} T_v \mod \mathscr{B}_n^{\rhd (f,\lambda)} \mid (\mathfrak{t}, v) \in I(f,\lambda)\,\}.$$

We need Enyang's basis for $\Delta(f,\lambda)$ which is indexed by up-down tableaux.

Given a $(f,\lambda) \in \Lambda_n$. An n–updown λ–tableau, or more simply an updown λ–tableau, is a sequence $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_n)$ of partitions such that $\mathfrak{t}_n = \lambda$, $\mathfrak{t}_0 = \varnothing$, and either $\mathfrak{t}_{i-1} \to \mathfrak{t}_i$ or $\mathfrak{t}_i \to \mathfrak{t}_{i-1}$ for $i = 1, \ldots, n$. Let $\mathscr{T}_n^{ud}(\lambda)$ be the set of updown λ–tableaux of n.

In what follows, we define $T_{i,j} = T_i T_{i+1} \cdots T_{j-1}$ (resp. $T_{i-1} T_{i-2} \cdots T_j$) if $j > i$ (resp. if $j < i$). If $i = j$, we set $T_{i,j} = 1$.

Definition 2.14. (cf. [7]) Given $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $\lambda \in \Lambda^+(n-2f)$, $0 \le f \le \lfloor \frac{n}{2} \rfloor$, define the non-negative integer f_j, $1 \le j \le n$ and $0 \le f_j \le \lfloor j/2 \rfloor$ by declaring that $\mathfrak{t}_j \vdash j - 2f_j$. Let $\mu^{(j)} = \mathfrak{t}_j$. Define $\mathfrak{M}_\mathfrak{t} = \mathfrak{M}_{\mathfrak{t}_n}$ inductively by declaring that

(1) $\mathfrak{M}_{\mathfrak{t}_1} = 1$,

(2) $\mathfrak{M}_{\mathfrak{t}_i} = \sum_{j=a_{k-1}+1}^{a_k} q^{a_k - j} T_{j,i} \mathfrak{M}_{\mathfrak{t}_{i-1}}$ if $\mathfrak{t}_i = \mathfrak{t}_{i-1} \cup p$ with $p = (k, \mu_k^{(i)})$, and $a_l = 2f_i + \sum_{j=1}^{\ell} \mu_j^{(i)}$

(3) $\mathfrak{M}_{\mathfrak{t}_i} = E_{2f_i - 1} T_{i,2f_i}^{-1} T_{b_k, 2f_i - 1}^{-1} \mathfrak{M}_{\mathfrak{t}_{i-1}}$ if $\mathfrak{t}_{i-1} = \mathfrak{t}_i \cup p$ with $p = (k, \mu_k^{(i-1)})$, and $b_k = 2(f_i - 1) + \sum_{j=1}^{k} \mu_j^{(i-1)}$.

It follows from the definition that $\mathfrak{M}_\mathfrak{t} = \mathfrak{M}_\lambda b_\mathfrak{t}$ for some $b_\mathfrak{t} \in \mathscr{B}_n$. The following recursive formula describe explicitly the element $b_\mathfrak{t}$. Note that $b_\mathfrak{t} = b_{\mathfrak{t}_n}$ and $\mathfrak{t}_{n-1} = \mu$.

$$b_{\mathfrak{t}_n} = \begin{cases} T_{a_k, n} b_{\mathfrak{t}_{n-1}}, & \text{if } \mathfrak{t}_n = \mathfrak{t}_{n-1} \cup \{(k, \lambda_k)\} \\ T_{n,2f}^{-1} \sum_{j=b_{k-1}+1}^{b_k} q^{b_k - j} T_{j,2f-1}^{-1} b_{\mathfrak{t}_{n-1}}, & \text{if } \mathfrak{t}_{n-1} = \mathfrak{t}_n \cup \{(k, \mu_k)\}. \end{cases} \tag{2.15}$$

Suppose $\lambda \in \Lambda^+(n-2f)$ with s removable nodes $p_1, p_2, \cdots, p_s$ and $m-s$ addable nodes $p_{s+1}, p_{s+2}, \ldots, p_m$.

- Let $\mu^{(i)} \in \Lambda^+(n-2f-1)$ be obtained from λ by removing the box p_i for $1 \le i \le s$.
- Let $\mu^{(j)} \in \Lambda^+(n-2f+1)$ be obtained from λ by adding the box p_j for $s+1 \le j \le m$.

We identify $\mu^{(i)}$ with $(k_i, \mu^{(i)}) \in \Lambda_{n-1}$ for $1 \le i \le m$. So, $\mu^{(i)} \rhd \mu^{(j)}$ for each i,j with $1 \le i \le s$ and $s+1 \le j \le m$, and $k_i = f$ if $1 \le i \le s$ and $f-1$ otherwise. We arrange $(k_i, \mu^{(i)})$'s such that $(k_1, \mu^{(1)}) \rhd (k_2, \mu^{(2)}) \cdots \rhd (k_m, \mu^{(m)})$.

Define

$$N^{\unrhd \mu^{(i)}} = R\text{–span}\{\mathfrak{M}_\mathfrak{t} \pmod{\mathscr{B}_n^{\rhd (f,\lambda)}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda), \mathfrak{t}_{n-1} \unrhd \mu^{(i)}\},$$
$$N^{\rhd \mu^{(i)}} = R\text{–span}\{\mathfrak{M}_\mathfrak{t} \pmod{\mathscr{B}_n^{\rhd (f,\lambda)}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda), \mathfrak{t}_{n-1} \rhd \mu^{(i)}\}.$$

In order to simplify the notation, we use $\overline{\mathfrak{M}}_\mathfrak{t}$ instead of $\mathfrak{M}_\mathfrak{t} \pmod{\mathscr{B}_n^{\rhd (f,\lambda)}}$ later on. The following result is due to Enyang.

Theorem 2.16. [7] *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over R. Assume that $(f,\lambda)\in\Lambda_n$.*

a) *$\{\overline{\mathfrak{M}}_{\mathfrak{t}} \mid \mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)\}$ is an R-basis of $\Delta(f,\lambda)$.*
b) *Both $N^{\trianglerighteq\mu^{(i)}}$ and $N^{\triangleright\mu^{(i)}}$ are $\mathscr{B}_{n-1}$-submodules of $\Delta(f,\lambda)$.*
c) *The R-linear map $\phi : N^{\trianglerighteq\mu^{(i)}}/N^{\triangleright\mu^{(i)}} \to \Delta(k_i,\mu^{(i)})$ sending $\mathfrak{M}_{\mathfrak{t}}$ $\pmod{N^{\triangleright\mu^{(i)}}}$ to $\mathfrak{M}_{\mathfrak{t}_{n-1}}$ $\pmod{\mathscr{B}_{n-1}^{\triangleright(k_i,\mu^{(i)})}}$ is an isomorphism of $\mathscr{B}_{n-1}$-modules.*

Definition 2.17. Given $\mathfrak{s},\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$, define $\mathfrak{M}_{\mathfrak{s},\mathfrak{t}}=b_{\mathfrak{s}}^*\mathfrak{M}_\lambda b_{\mathfrak{t}}$ where $*:\mathscr{B}_n\to\mathscr{B}_n$ is the R-linear anti-involution on $\mathscr{B}_n$ defined in Lemma 2.2.

Standard arguments prove the following result (cf. [18, Theorem 2.7]).

Corollary 2.18. *Suppose that $\mathscr{B}_n$ is the Birman-Wenzl algebra over R. Then*

a) *$\mathscr{M}_n=\{\mathfrak{M}_{\mathfrak{s}\mathfrak{t}}\mid \mathfrak{s},\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda),\lambda\vdash n-2f, 0\le f\le\lfloor\frac{n}{2}\rfloor\}$ is a free R–basis of $\mathscr{B}_n$.*
b) *$\mathfrak{M}_{\mathfrak{s}\mathfrak{t}}^*=\mathfrak{M}_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{s},\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ and all $(f,\lambda)\in\Lambda_n$.*
c) *Let $\widetilde{\mathscr{B}_n}^{\triangleright(f,\lambda)}$ be the free R-submodule of $\mathscr{B}_n$ generated by $\mathfrak{M}_{\tilde{\mathfrak{s}}\tilde{\mathfrak{t}}}$ with $\tilde{\mathfrak{s}},\tilde{\mathfrak{t}}\in\mathscr{T}_n^{ud}(\mu)$ and $(\frac{n-|\mu|}{2},\mu)\triangleright(f,\lambda)$. Then $\widetilde{\mathscr{B}_n}^{\triangleright(f,\lambda)}=\mathscr{B}_n^{\triangleright(f,\lambda)}$.*
d) *For all $\mathfrak{s},\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$, and all $h\in\mathscr{B}_n$, there exist scalars $a_{\mathfrak{u}}\in R$ which are independent of $\mathfrak{s}$, such that*
$$\mathfrak{M}_{\mathfrak{s}\mathfrak{t}}h\equiv\sum_{\mathfrak{u}}a_{\mathfrak{u}}\mathfrak{M}_{\mathfrak{s}\mathfrak{u}}\pmod{{\mathscr{B}_n}^{\triangleright(f,\lambda)}}.$$

We call $\mathscr{M}_n$ the Jucys-Murphy basis of $\mathscr{B}_n$. It is a cellular basis of $\mathscr{B}_n$ over R. In [8], Graham and Lehrer proved that there is a symmetric invariant bilinear form $\langle\ ,\ \rangle:\Delta(f,\lambda)\times\Delta(f,\lambda)\to R$ on each cell module. In our case, we use $\mathscr{M}_n$ to define such a bilinear form on $\Delta(f,\lambda)$. More explicitly, $\langle\overline{\mathfrak{M}}_{\mathfrak{s}},\overline{\mathfrak{M}}_{\mathfrak{t}}\rangle\in R$ is determined by
$$\mathfrak{M}_{\tilde{\mathfrak{s}}\mathfrak{s}}\mathfrak{M}_{\mathfrak{t}\tilde{\mathfrak{t}}}\equiv\langle\overline{\mathfrak{M}}_{\mathfrak{s}},\overline{\mathfrak{M}}_{\mathfrak{t}}\rangle\mathfrak{M}_{\tilde{\mathfrak{s}}\tilde{\mathfrak{t}}}\pmod{\mathscr{B}_n^{\triangleright(f,\lambda)}},\quad \tilde{\mathfrak{s}},\tilde{\mathfrak{t}}\in\mathscr{T}_n^{ud}(\lambda).$$

By Corollary 2.18(d), the above symmetric invariant bilinear form is independent of $\tilde{\mathfrak{s}},\tilde{\mathfrak{t}}\in\mathscr{T}_n^{ud}(\lambda)$. The Gram matrix $G_{f,\lambda}$ with respect to the Jucys-Murphy basis of $\Delta(f,\lambda)$ is the $k\times k$ matrix with
$$k=\operatorname{rank}\Delta(f,\lambda)=\frac{n!(2f-1)!!}{(2f)!\prod_{(i,j)\in\lambda}h_{i,j}^\lambda}$$
where $h_{i,j}^\lambda=\lambda_i+\lambda_j'-i-j+1$ is the hook length. The $(\mathfrak{s},\mathfrak{t})$-th entry of $G_{f,\lambda}$ is $\langle\overline{\mathfrak{M}}_{\mathfrak{s}},\overline{\mathfrak{M}}_{\mathfrak{t}}\rangle$.

One of the main purposes of this paper is to compute the Gram determinant $\det G_{f,\lambda}$ associated to each cell module $\Delta(f,\lambda)$.

Given $\mathfrak{s}\in\mathscr{T}_n^{ud}(\lambda)$, we identify $\mathfrak{s}_i$ with $(f_i,\mu^{(i)})$ if $\mathfrak{s}_i=\mu^{(i)}\vdash i-2f_i$. Define the partial order $\trianglelefteq$ on $\mathscr{T}_n^{ud}(\lambda)$ by declaring that $\mathfrak{s}\trianglelefteq\mathfrak{t}$ if $\mathfrak{s}_i\trianglelefteq\mathfrak{t}_i$ for all $1\le i\le n$. Write $\mathfrak{s}\triangleleft\mathfrak{t}$ if $\mathfrak{s}\trianglelefteq\mathfrak{t}$ and $\mathfrak{s}\ne\mathfrak{t}$. We remark that Enyang has used $\trianglelefteq$ to state Theorem 2.19. We define the relation $\succ$ on $\mathscr{T}_n^{ud}(\lambda)$ instead of his partial order $\trianglelefteq$.

Suppose $\mathfrak{s} \neq \mathfrak{t}$. We write $\mathfrak{s} \succ \mathfrak{t}$ if there is a positive integer $k \le n-1$ such that $\mathfrak{s}_k \rhd \mathfrak{t}_k$ and $\mathfrak{s}_j = \mathfrak{t}_j$ for $k+1 \le j \le n$. We will use $\mathfrak{s} \overset{k}{\succ} \mathfrak{t}$ to denote $\mathfrak{s}_j \rhd \mathfrak{t}_j$ and $\mathfrak{s}_\ell = \mathfrak{t}_\ell$ for $j+1 \le \ell \le n$ and $j \ge k$.

For any $(f,\lambda) \in \Lambda_n$, define $\mathfrak{t}^\lambda \in \mathscr{T}_n^{ud}(\lambda)$ such that

- $\mathfrak{t}^\lambda_{2i-1} = (1)$ and $\mathfrak{t}^\lambda_{2i} = \varnothing$ for $1 \le i \le f$,
- $\mathfrak{t}^\lambda_i$ is obtained from $\hat{\mathfrak{t}}^\lambda$ by removing the entries j with $j > i$ under the assumption $2f+1 \le i \le n$.

Then $\mathfrak{t}^\lambda$ is maximal in $\mathscr{T}_n^{ud}(\lambda)$ with respect to $\succ$ and $\unrhd$.

For any $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda) \in \Lambda_n$, define $c_\mathfrak{t}(k) \in R$ by

$$c_\mathfrak{t}(k) = \begin{cases} rq^{2(j-i)}, & \text{if } \mathfrak{t}_k = \mathfrak{t}_{k-1} \cup (i,j), \\ r^{-1}q^{2(i-j)}, & \text{if } \mathfrak{t}_{k-1} = \mathfrak{t}_k \cup (i,j). \end{cases}$$

If $p = (i,j)$ is an addable (resp. a removable) node of λ, define $c_\lambda(p) = j - i$ (resp. $-j+i$).

The following result plays a key role in the construction of an orthogonal basis for $\mathscr{B}_n$.

Theorem 2.19. [7] *Given $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$, with $(f,\lambda) \in \Lambda_n$,*

$$\mathfrak{M}_{\mathfrak{s}\mathfrak{t}} L_k \equiv c_\mathfrak{t}(k)\mathfrak{M}_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{u} \overset{k-1}{\succ} \mathfrak{t}} a_\mathfrak{u} \mathfrak{M}_{\mathfrak{s}\mathfrak{u}} \pmod{\mathscr{B}_n^{\rhd(f,\lambda)}}.$$

Enyang used $\mathfrak{u} \rhd \mathfrak{t}$ instead of $\mathfrak{u} \succ \mathfrak{t}$. However, we could not understand the claim about $\breve{N}^\mu$ under [7, (7.3)]. If one uses $\succ$ instead of $\rhd$, then everything in the proof of [7, 7.8] is available.

3. ORTHOGONAL REPRESENTATIONS FOR $\mathscr{B}_n$

In this section, we assume that F is a field which contains non-zero q, r and $(q-q^{-1})^{-1}$ such that $o(q^2) > n$ and $|c| > 2n-3$ whenever $r^2q^{2c} = 1$ for some $c \in \mathbb{Z}$. The main purpose of this section is to construct an orthogonal basis of $\mathscr{B}_n$ over F.

Suppose $1 \le k \le n$ and $(f,\lambda) \in \Lambda_n$. Define an equivalence relation $\overset{k}{\sim}$ on $\mathscr{T}_n^{ud}(\lambda)$ by declaring that $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$ if $\mathfrak{t}_j = \mathfrak{s}_j$ whenever $1 \le j \le n$ and $j \neq k$, for $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. The following result is well-known. See e.g. [18].

Lemma 3.1. *Suppose $s \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$. Then there is a bijection between the set of all addable and removable nodes of $\mathfrak{s}_{k+1}$ and the set $\{\, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda) \mid \mathfrak{t} \overset{k}{\sim} \mathfrak{s} \,\}$.*

Suppose λ and μ are partitions. We write $\lambda \ominus \mu = \alpha$ if either $\lambda \supset \mu$ and $\lambda \setminus \mu = \alpha$ or $\lambda \subset \mu$ and $\mu \setminus \lambda = \alpha$. The following lemma can be proved by arguments similar to those in [18].

Lemma 3.2. *Assume that $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda) \in \Lambda_n$.*

a) $\mathfrak{s} = \mathfrak{t}$ *if and only if* $c_\mathfrak{s}(k) = c_\mathfrak{t}(k)$ *for* $1 \le k \le n$.

b) *Suppose* $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$. *Then* $c_\mathfrak{t}(k) \neq c_\mathfrak{t}(k+1)$.

c) If $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$, then $c_{\mathfrak{t}}(k) \neq c_{\mathfrak{s}}(k)^{\pm}$ whenever $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$.
d) $c_{\mathfrak{t}}(k) \notin \{-q, q^{-1}\}$ for all $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$.

Definition 3.3. Suppose $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ for some $(f, \lambda) \in \Lambda_n$. Following [13], we define

a) $\mathscr{R}(k) = \{ c_{\mathfrak{s}}(k) \mid \mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda), (f, \lambda) \in \Lambda_n \}, \quad 1 \le k \le n,$
b) $F_{\mathfrak{t}} = \prod_{k=1}^{n} F_{\mathfrak{t},k}$ where $F_{\mathfrak{t},k} = \prod_{\substack{r \in \mathscr{R}(k) \\ c_{\mathfrak{t}}(k) \neq r}} \frac{L_k - r}{c_{\mathfrak{t}}(k) - r}$, and $f_{\mathfrak{t}} = \overline{\mathfrak{M}}_{\mathfrak{t}} F_{\mathfrak{t}}, \quad \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$,
c) $f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} \mathfrak{M}_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}, \quad \mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$.

Standard arguments prove Lemma 3.4 and Lemma 3.5. Mathas has proved similar results for a general class of cellular algebras in [14]. Although he has used a partial order which is similar to $\trianglelefteq$, his arguments can be used to verify the following results. See also [12] for the Hecke algebra $\mathscr{H}_n$ of type A.

Lemma 3.4. *Suppose that $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_n$.*

a) $f_{\mathfrak{t}} = \overline{\mathfrak{M}}_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} a_{\mathfrak{s}} \overline{\mathfrak{M}}_{\mathfrak{s}}$.
b) $\overline{\mathfrak{M}}_{\mathfrak{t}} = f_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} b_{\mathfrak{s}} f_{\mathfrak{s}}$.
c) $f_{\mathfrak{t}} L_k = c_{\mathfrak{t}}(k) f_{\mathfrak{t}}$, for any k, $1 \le k \le n$.
d) $f_{\mathfrak{t}} F_{\mathfrak{s}} = \delta_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{t}}$ for all $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ with $(\frac{n-|\mu|}{2}, \mu) \in \Lambda_n$.
e) $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)\}$ is a basis of $\Delta(f, \lambda)$.
f) The Gram determinants associated to $\Delta(f, \lambda)$ defined by $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)\}$ and $\{\overline{\mathfrak{M}}_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)\}$ are the same.

Let $f_{\mathfrak{t}} T_k = \sum_{\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)} s_{\mathfrak{t}\mathfrak{s}}(k) f_{\mathfrak{s}}$ and $f_{\mathfrak{t}} E_k = \sum_{\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)} E_{\mathfrak{t}\mathfrak{s}}(k) f_{\mathfrak{s}}$.

Lemma 3.5. *Suppose $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k \le n-1$.*

a) $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$ if either $s_{\mathfrak{t}\mathfrak{s}}(k) \neq 0$ or $E_{\mathfrak{t}\mathfrak{s}}(k) \neq 0$.
b) $f_{\mathfrak{t}} E_k = 0$ if $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$.
c) If $\mathfrak{t}_k \ominus \mathfrak{t}_{k-1}$ and $\mathfrak{t}_{k+1} \ominus \mathfrak{t}_k$ are neither in the same row nor in the same column, then there is a unique up-down tableau in $\mathscr{T}_n^{ud}(\lambda)$, denoted by $\mathfrak{t}s_k$, such that $\mathfrak{t}s_k \overset{k}{\sim} \mathfrak{t}$ and $c_{\mathfrak{t}}(k) = c_{\mathfrak{t}s_k}(k+1)$ and $c_{\mathfrak{t}}(k+1) = c_{\mathfrak{t}s_k}(k)$.
d) If $\mathfrak{t}_k \ominus \mathfrak{t}_{k-1}$ and $\mathfrak{t}_{k+1} \ominus \mathfrak{t}_k$ are either in the same row or in the same column, then there is no $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ such that $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$ and $c_{\mathfrak{t}}(k) = c_{\mathfrak{s}}(k+1)$ and $c_{\mathfrak{t}}(k+1) = c_{\mathfrak{s}}(k)$.

Lemma 3.6. *Suppose that $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{i-2} \neq \mathfrak{t}_i$, $\mathfrak{t}s_{i-1} \in \mathscr{T}_n^{ud}(\lambda)$ and $\mathfrak{t}s_{i-1} \lhd \mathfrak{t}$. We have*

a) *If $\mathfrak{t}_{i-2} \subset \mathfrak{t}_{i-1} \subset \mathfrak{t}_i$, then $\overline{\mathfrak{M}}_{\mathfrak{t}} T_{i-1} = \overline{\mathfrak{M}}_{\mathfrak{t}s_{i-1}}$.*
b) *If $\mathfrak{t}_{i-2} \supset \mathfrak{t}_{i-1} \subset \mathfrak{t}_i$ such that $\ell > k$ where $\mathfrak{t}_{i-2} \backslash \mathfrak{t}_{i-1} = (k, \nu_k)$, $\mathfrak{t}_i \backslash \mathfrak{t}_{i-1} = (\ell, \mu_\ell)$, $\mathfrak{t}_{i-2} = \nu$ and $\mathfrak{t}_i = \mu$, then $\overline{\mathfrak{M}}_{\mathfrak{t}} T_{i-1}^{-1} = \overline{\mathfrak{M}}_{\mathfrak{t}s_{i-1}}$.*

Proof. The proof of the result is essentially identical to the proof of the corresponding result in the proof of [18, 3.14]. One can check it by Definitions 1.1 and 2.14. We leave the details to the reader. □

Lemma 3.7. *Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\mathfrak{t}_{i-1} \neq \mathfrak{t}_{i+1}$ *and* $\mathfrak{t}s_i \in \mathscr{T}_n^{ud}(\lambda)$. *Then* $f_\mathfrak{t} T_i = s_{\mathfrak{t}\mathfrak{t}}(i) f_\mathfrak{t} + s_{\mathfrak{t},\mathfrak{t}s_i}(i) f_{\mathfrak{t}s_i}$, *where*

- $s_{\mathfrak{t}\mathfrak{t}}(i) = \frac{\omega c_\mathfrak{t}(i+1)}{c_\mathfrak{t}(i+1) - c_\mathfrak{t}(i)}$,
- $s_{\mathfrak{t},\mathfrak{t}s_i}(i) = 1 - \frac{c_\mathfrak{t}(i)}{c_\mathfrak{t}(i+1)} s_{\mathfrak{t}\mathfrak{t}}^2(i)$ *if* $\mathfrak{t}s_i \rhd \mathfrak{t}$ *and* $s_{\mathfrak{t},\mathfrak{t}s_i}(i) = 1$ *if* $\mathfrak{t}s_i \lhd \mathfrak{t}$ *and one of the following conditions holds,*
  - *a)* $\mathfrak{t}_{i-1} \subset \mathfrak{t}_i \subset \mathfrak{t}_{i+1}$,
  - *b)* $\mathfrak{t}_{i-1} \supset \mathfrak{t}_i \subset \mathfrak{t}_{i+1}$ *such that* $\ell > k$ *where* $\mathfrak{t}_{i-1} \setminus \mathfrak{t}_i = (k, \nu_k)$, $\mathfrak{t}_{i+1} \setminus \mathfrak{t}_i = (\ell, \mu_\ell)$, $\mathfrak{t}_{i-1} = \nu$ *and* $\mathfrak{t}_{i+1} = \mu$.

Proof. Write $f_\mathfrak{t} T_i = \sum_{\mathfrak{s} \overset{i}{\sim} \mathfrak{t}} s_{\mathfrak{t}\mathfrak{s}}(i) f_\mathfrak{s}$. By Lemma 3.5, $s_{\mathfrak{t}\mathfrak{s}}(i) \neq 0$ implies $\mathfrak{s} \in \{\mathfrak{t}, \mathfrak{t}s_i\}$, and $f_\mathfrak{t} E_i = 0$. On the other hand, by Lemma 2.1(c),

$$T_i L_{i+1} = L_i T_i + \omega L_{i+1} - \omega r^{-1} E_i L_i T_i.$$

So, $f_\mathfrak{t} T_i L_{i+1} = c_\mathfrak{t}(i) f_\mathfrak{t} T_i + \omega c_\mathfrak{t}(i+1) f_\mathfrak{t}$. Comparing the coefficient of $f_\mathfrak{t}$ in $f_\mathfrak{t} T_i L_{i+1}$ yields the formula on $s_{\mathfrak{t}\mathfrak{t}}(i)$ as required.

We compute $s_{\mathfrak{t},\mathfrak{t}s_i}(i)$ under the assumptions as follows. By Lemma 3.6, $\overline{\mathfrak{M}}_\mathfrak{t} T_i = \overline{\mathfrak{M}}_{\mathfrak{t}s_i}$ if $\mathfrak{t}_{i-1} \subset \mathfrak{t}_i \subset \mathfrak{t}_{i+1}$ and $\mathfrak{t}s_i \lhd \mathfrak{t}$. By Lemma 3.4(a)–(b),

$$f_\mathfrak{t} T_i = (\overline{\mathfrak{M}}_\mathfrak{t} + \sum_{\mathfrak{u} \succ \mathfrak{t}} a_\mathfrak{u} f_\mathfrak{u}) T_i = \overline{\mathfrak{M}}_{\mathfrak{t}s_i} + \sum_{\mathfrak{u} \succ \mathfrak{t}} a_\mathfrak{u} f_\mathfrak{u} T_i.$$

If $f_{\mathfrak{t}s_i}$ appears in the expression of $f_\mathfrak{u} T_i$ with non-zero coefficient, then $\mathfrak{u} \overset{i}{\sim} \mathfrak{t}s_i$. Therefore, $\mathfrak{u} \in \{\mathfrak{t}, \mathfrak{t}s_i\}$ which contradicts $\mathfrak{u} \succ \mathfrak{t} \rhd \mathfrak{t}s_i$. By Lemma 3.4(a), the coefficient of $f_{\mathfrak{t}s_i}$ in $f_\mathfrak{t} T_i$ is 1.

Under the assumption given in (b), we have $\overline{\mathfrak{M}}_\mathfrak{t} T_i^{-1} = \overline{\mathfrak{M}}_{\mathfrak{t}s_i}$. Thus,

$$\begin{aligned} \overline{\mathfrak{M}}_\mathfrak{t} T_i &= \overline{\mathfrak{M}}_{\mathfrak{t}s_i} T_i^2 = \overline{\mathfrak{M}}_{\mathfrak{t}s_i}(1 + \omega(T_i - r^{-1}E_i)) \\ &= \overline{\mathfrak{M}}_{\mathfrak{t}s_i} + \omega \overline{\mathfrak{M}}_\mathfrak{t} - \omega r^{-1} \overline{\mathfrak{M}}_{\mathfrak{t}s_i} E_i \end{aligned}$$

By Lemma 3.4, $\overline{\mathfrak{M}}_{\mathfrak{t}s_i} = f_{\mathfrak{t}s_i} + \sum_{\mathfrak{u} \succ \mathfrak{t}s_i} a_\mathfrak{u} f_\mathfrak{u}$ for some $a_\mathfrak{u} \in F$. Since $\mathfrak{t}s_i \overset{i}{\sim} \mathfrak{t}$, $(\mathfrak{t}s_i)_{i-1} \neq (\mathfrak{t}s_i)_{i+1}$. By Lemma 3.5(b), $f_{\mathfrak{t}s_i} E_i = 0$. If $f_{\mathfrak{t}s_i}$ appears in the expression of $f_\mathfrak{u} E_i$ with non-zero coefficient, then $\mathfrak{u} \overset{i}{\sim} \mathfrak{t}s_i$, forcing $\mathfrak{u}_{i-1} \neq \mathfrak{u}_{i+1}$. Thus, $f_\mathfrak{u} E_i = 0$, a contradiction. Finally, by Lemma 3.4(a), the coefficient of $f_{\mathfrak{t}s_i}$ in $\overline{\mathfrak{M}}_{\mathfrak{t}s_i}$ is 1, forcing $s_{\mathfrak{t},\mathfrak{t}s_i}(i) = 1$. □

Note that the bilinear form $\langle\ ,\ \rangle : \Delta(f,\lambda) \times \Delta(f,\lambda) \to F$ is associative. We have $\langle f_\mathfrak{t} T_k, f_\mathfrak{t} T_k \rangle = \langle f_\mathfrak{t} T_k^2, f_\mathfrak{t} \rangle$. The proofs of Corollary 3.8 and Lemma 3.9 are essentially identical to [18, 4.3, 3.15]. We leave the details to the reader.

Corollary 3.8. *Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $(f,\lambda) \in \Lambda_n$ *and* $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$. *If* $\mathfrak{t}s_k \in \mathscr{T}_n^{ud}(\lambda)$ *and* $\mathfrak{t}s_k \lhd \mathfrak{t}$, *then*

$$\langle f_{\mathfrak{t}s_k}, f_{\mathfrak{t}s_k} \rangle = \left(1 - \frac{\omega^2 c_\mathfrak{t}(k) c_\mathfrak{t}(k+1)}{(c_\mathfrak{t}(k+1) - c_\mathfrak{t}(k))^2}\right) \langle f_\mathfrak{t}, f_\mathfrak{t} \rangle.$$

Lemma 3.9. *Suppose* $(f,\lambda) \in \Lambda_n$. *If* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$, *then*

a) $f_\mathfrak{t} T_k = q f_\mathfrak{t}$ *if* $\mathfrak{t}_k \ominus \mathfrak{t}_{k-1}$ *and* $\mathfrak{t}_k \ominus \mathfrak{t}_{k+1}$ *are in the same row,*

b) $f_\mathfrak{t} T_k = -q^{-1} f_\mathfrak{t}$ *if* $\mathfrak{t}_k \ominus \mathfrak{t}_{k-1}$ *and* $\mathfrak{t}_k \ominus \mathfrak{t}_{k+1}$ *are in the same column.*

In the following, we assume that $F = \mathbb{C}(r^{\pm}, q^{\pm}, \omega^{-1})$, where r, q are indeterminates and $\omega = q - q^{-1}$.

Lemma 3.10. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and* $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$. *Then*

a) $E_{\mathfrak{t}\mathfrak{t}}(k) = rc_{\mathfrak{t}}(k)^{-1}(1 + \omega^{-1}(c_{\mathfrak{t}}(k) - c_{\mathfrak{t}}(k)^{-1}) \prod_{\substack{\mathfrak{s} \overset{k}{\sim} \mathfrak{t} \\ \mathfrak{s} \neq \mathfrak{t}}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)^{-1}}{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)} \neq 0.$

b) $E_{\mathfrak{t}\mathfrak{s}}(k)E_{\mathfrak{u}\mathfrak{u}}(k) = E_{\mathfrak{t}\mathfrak{u}}(k)E_{\mathfrak{u}\mathfrak{s}}(k)$ *for any* $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\mathfrak{t} \overset{k}{\sim} \mathfrak{s} \overset{k}{\sim} \mathfrak{u}$.

Proof. First, we prove $E_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$. By assumption and [21, 5.6], $\Delta(f, \lambda)$ is irreducible since $\mathscr{B}_n$ is semisimple. In [11, 6.17], Leduc and Ram proved that the seminormal representations $S^{f,\lambda}$[2] over $\mathbb{C}$ with special parameters for all $(f, \lambda) \in \Lambda_n$ consist of the complete set of pair-wise non-isomorphic irreducible modules when $\mathscr{B}_n$ is semisimple. By the fundamental theorem of algebra, one can get the same results over $\mathbb{C}(q^{\pm}, r^{\pm}, \omega^{-1})$ where r, q are indeterminates and $\omega = q - q^{-1}$.

Thus, $\Delta(f, \lambda) \cong S^{\ell,\mu}$ for some $(\ell, \mu) \in \Lambda_n$. If we denote by ϕ the corresponding isomorphism between $\Delta(f, \lambda)$ and $S^{\ell,\mu}$, then $\phi(f_{\mathfrak{t}}) \in S^{\ell,\mu}$. In [11], Leduc and Ram constructed a basis for $S^{\ell,\mu}$, say $v_{\mathfrak{s}}$, $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ such that $v_{\mathfrak{s}} L_k = c_{\mathfrak{s}}(k) v_{\mathfrak{s}}$. Note that $f_{\mathfrak{t}} \in \Delta(f, \lambda)$ is a common eigenvector of L_k, $1 \le k \le n$. By Lemma 3.2(a), $(\ell, \mu) = (f, \lambda)$ and $\phi(f_{\mathfrak{t}})$ is equal to $v_{\mathfrak{t}}$ up to a scalar since the common eigenspace on which $L_k, 1 \le k \le n$ acts as $c_{\mathfrak{s}}(k)$ is of one dimension. Leduc and Ram [11, 5.9][3] proved that $\tilde{E}_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$ for any $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$, where $\tilde{E}_{\mathfrak{t}\mathfrak{s}}(k)$ is defined by $v_{\mathfrak{t}} E_k = \sum_{\mathfrak{s} \overset{k}{\sim} \mathfrak{t}} \tilde{E}_{\mathfrak{t}\mathfrak{s}}(k) v_{\mathfrak{s}}$. Since $v_{\mathfrak{t}}$ is a non-zero scalar of $\phi(f_{\mathfrak{t}})$, $E_{\mathfrak{t}\mathfrak{t}}(k) = \tilde{E}_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$. We remark that $E_{\mathfrak{t}\mathfrak{t}}(k) \in F$ since $f_{\mathfrak{t}}$ is an F-basis element of $\Delta(f, \lambda)$. In general, $E_{\mathfrak{t}\mathfrak{s}}(k) \neq \tilde{E}_{\mathfrak{t}\mathfrak{s}}(k)$ if $\mathfrak{s} \neq \mathfrak{t}$.

In [2], Beliakova and Blanchet proved that the generating function $W_k(y) = \sum_{a \ge 0} \omega_k^{(a)}/y^a$ satisfies the following identity

$$\frac{W_{k+1}(y) + r^{-1}\omega^{-1} - \frac{y^2}{y^2-1}}{W_k(y) + r^{-1}\omega^{-1} - \frac{y^2}{y^2-1}} = \frac{y^{-1} - \frac{\omega^2 L_k^{-1}}{(y - L_k^{-1})^2}}{y^{-1} - \frac{\omega^2 L_k}{(y - L_k)^2}}.$$

Comparing the coefficients of $f_{\mathfrak{s}}$ on both sides of $f_{\mathfrak{t}} E_k W_k(y) = f_{\mathfrak{t}} E_k \frac{y}{y - L_k} E_k$ yields

$$W_k(y, \mathfrak{s}) y^{-1} E_{\mathfrak{t}\mathfrak{s}}(k) = \sum_{\mathfrak{s} \overset{k}{\sim} \mathfrak{t} \overset{k}{\sim} \mathfrak{u}} \frac{E_{\mathfrak{t}\mathfrak{u}}(k) E_{\mathfrak{u}\mathfrak{s}}(k)}{y - c_{\mathfrak{u}}(k)}.$$

Thus $E_{\mathfrak{t}\mathfrak{s}}(k) \cdot Res_{y = c_{\mathfrak{u}}(k)} W_k(y, \mathfrak{s}) y^{-1} = E_{\mathfrak{u}\mathfrak{s}}(k) E_{\mathfrak{t}\mathfrak{u}}(k)$. Since $E_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$, $E_{\mathfrak{t}\mathfrak{t}}(k) = Res_{y = c_{\mathfrak{t}}(k)} W_k(y, \mathfrak{t}) y^{-1}$ by assuming that $\mathfrak{t} = \mathfrak{s} = \mathfrak{u}$. By computation, we can verify

$$\operatorname*{Res}_{y = c_{\mathfrak{t}}(k)} \frac{W_k(y, \mathfrak{t})}{y} = rc_{\mathfrak{t}}(k)^{-1}(1 + \omega^{-1}(c_{\mathfrak{t}}(k) - c_{\mathfrak{t}}(k)^{-1})) \prod_{\substack{\mathfrak{s} \overset{k}{\sim} \mathfrak{t} \\ \mathfrak{s} \neq \mathfrak{t}}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)^{-1}}{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)}.$$

This completes the proof of (a). If $\mathfrak{s} \overset{k}{\sim} \mathfrak{u}$, then $c_{\mathfrak{s}}(j) = c_{\mathfrak{u}}(j)$ for $j \le k - 1$. By Lemma 2.5, $\omega_k^{(a)} \in F[L_1^{\pm}, L_2^{\pm}, \cdots, L_{k-1}^{\pm}]$. So, $W_k(y, \mathfrak{s}) = W_k(y, \mathfrak{u})$. Thus $E_{\mathfrak{t}\mathfrak{s}}(k) E_{\mathfrak{u}\mathfrak{u}}(k) = E_{\mathfrak{u}\mathfrak{s}}(k) E_{\mathfrak{t}\mathfrak{u}}(k)$, proving (b). □

[2] In [11], $S^{f,\lambda}$ is denoted by $\mathscr{Z}^{\lambda}$

[3] In [11], $r = \varepsilon q^a$ for some $a \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$. Therefore, $\tilde{E}_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$ if r is an indeterminant.

The following result is a special case of [14, 3.14] which is about the construction of the primitive idempotents and central primitive idempotents for a general class of cellular algebras. In our case, such idempotents can be computed explicitly via a recursive formula on $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle$, $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ and $(f,\lambda)\in\Lambda_n$. This recovers the main result in [2].

Proposition 3.11. *Suppose that* $\mathscr{B}_n$ *is the Birman-Wenzl algebra over* $\mathbb{C}(q^{\pm}, r^{\pm}, \omega^{-1})$.

a) *Suppose* $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$. *Then* $\frac{1}{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle} f_{\mathfrak{t}\mathfrak{t}}$ *is a primitive idempotent of semisimple* $\mathscr{B}_n$ *with respect to the cell module* $\Delta(f,\lambda)$.
b) $\sum_{\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)} \frac{1}{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle} f_{\mathfrak{t}\mathfrak{t}}$ *is a central primitive idempotent. Furthermore,*

$$\sum_{(\frac{n-|\lambda|}{2},\lambda)\in\Lambda_n}\ \sum_{\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)} \frac{1}{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle} f_{\mathfrak{t}\mathfrak{t}} = 1.$$

4. Gram determinants for $\mathscr{B}_n$

In this section, we compute the Gram determinant for each cell module of $\mathscr{B}_n$ over $F=\mathbb{C}(q^{\pm}, r^{\pm}, \omega^{-1})$, where r, q are indeterminates and $\omega = q-q^{-1}$. Our result for the Gram determinants holds true for $\mathscr{B}_n$ over $R:=\mathbb{Z}[q^{\pm}, r^{\pm}, \omega^{-1}]$ since the Jucys-Murphy basis for $\Delta(f,\lambda)$ is an R-basis. By base change, it holds over an arbitrary field.

Given $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{n-1}=\mu$, define $\hat{\mathfrak{t}}\in\mathscr{T}_{n-1}^{ud}(\mu)$ such that $\hat{\mathfrak{t}}_i=\mathfrak{t}_i$, $1\le i\le n-1$, and $\tilde{\mathfrak{t}}\in\mathscr{T}_n^{ud}(\lambda)$ with $\tilde{\mathfrak{t}}_j=\mathfrak{t}^{\mu}_j$ for $1\le j\le n-1$ and $\tilde{\mathfrak{t}}_n=\mathfrak{t}_n=\lambda$.

Let $[n]=1+q^2+\cdots+q^{2n-2}$ and $[n]!=[n][n-1]\cdots[2][1]$. If $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)$, define $[\lambda]!=\prod_{i=1}^k[\lambda_i]!$. Standard arguments prove the following result (cf.[18, 4.2]). We leave the details to the reader.

Proposition 4.1. *Suppose* $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ *with* $(f,\lambda)\in\Lambda_n$. *If* $\mathfrak{t}_{n-1}=\mu$ *with* $(\ell,\mu)\in\Lambda_{n-1}$, *then* $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle = \langle f_{\hat{\mathfrak{t}}}, f_{\hat{\mathfrak{t}}}\rangle \frac{\langle f_{\tilde{\mathfrak{t}}}, f_{\tilde{\mathfrak{t}}}\rangle}{\delta^{\ell}[\mu]!}$.

For any $\lambda\vdash n-2f$, let $\mathscr{A}(\lambda)$ (resp. $\mathscr{R}(\lambda)$) be the set of all addable (resp. removable) nodes of λ. Given a $p=(k,\lambda_k)\in\mathscr{R}(\lambda)$ (resp. $p=(k,\lambda_k+1)\in\mathscr{A}(\lambda)$, define

a) $\mathscr{R}(\lambda)^{<p}=\{(\ell,\lambda_\ell)\in\mathscr{R}(\lambda)\mid \ell>k\}$,
b) $\mathscr{A}(\lambda)^{<p}=\{(\ell,\lambda_\ell+1)\in\mathscr{A}(\lambda)\mid \ell>k\}$,
c) $\mathscr{R}(\lambda)^{\ge p}=\{(\ell,\lambda_\ell)\in\mathscr{R}(\lambda)\mid \ell\le k\}$,
d) $\mathscr{A}(\lambda)^{\ge p}=\{(\ell,\lambda_\ell+1)\in\mathscr{A}(\lambda)\mid \ell\le k\}$.

Proposition 4.2. *Suppose* $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ *with* $(f,\lambda)\in\Lambda_n$. *If* $\hat{\mathfrak{t}}=\mathfrak{t}^{\mu}$ *and* $\mathfrak{t}_n=\mathfrak{t}_{n-1}\cup\{p\}$ *with* $p=(k,\lambda_k)$, *then*

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle}{\delta^f[\mu]!} = -q^{2\lambda_k}\frac{\prod_{r_1\in\mathscr{A}(\lambda)^{<p}}[c_\lambda(p)+c_\lambda(r_1)]}{\prod_{r_2\in\mathscr{R}(\lambda)^{<p}}[c_\lambda(p)-c_\lambda(r_2)]}. \tag{4.3}$$

Proof. By assumption, $\mathfrak{t} = \mathfrak{t}^\lambda s_{a,n}$ where $a = 2f + \sum_{j=1}^k \lambda_j$. Note that $\mathfrak{t} \lhd \mathfrak{t}s_{n-1} \lhd \cdots \lhd \mathfrak{t}s_{n,a}$. By Proposition 4.1 and Corollary 3.8,

$$\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle = \langle f_{\mathfrak{t}^\lambda}, f_{\mathfrak{t}^\lambda}\rangle \prod_{j=a+1}^{n} \left(1 - \omega^2 \frac{c_{\mathfrak{t}^\lambda}(j) c_{\mathfrak{t}^\lambda}(a)}{(c_{\mathfrak{t}^\lambda}(j) - c_{\mathfrak{t}^\lambda}(a))^2}\right). \tag{4.4}$$

Since $f_{\mathfrak{t}^\lambda} = \overline{\mathfrak{M}}_{\mathfrak{t}^\lambda}$, $\langle f_{\mathfrak{t}^\lambda}, f_{\mathfrak{t}^\lambda}\rangle = \delta^f [\lambda]!$. Using the definitions of $c_{\mathfrak{t}^\lambda}(j)$ for $a \le j \le n$ to simplify (4.4) yields (4.3). □

Proposition 4.5. *Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n - 2f$. *If* $\mathfrak{t}^\mu = \hat{\mathfrak{t}}$ *and* $\mathfrak{t}_{n-1} = \mathfrak{t}_n \cup p$ *with* $p = (k, \mu_k)$, *then*

$$\frac{\langle f_\mathfrak{t},\ f_\mathfrak{t}\rangle}{\delta^{f-1}[\mu]!} = [\mu_k] E_{\mathfrak{t}\mathfrak{t}}(n-1). \tag{4.6}$$

Proof. Write $\widetilde{\mathfrak{M}}_\lambda = \mathfrak{M}_{\mathfrak{t}^\lambda}$ with $\mathfrak{t}^\lambda \in \mathscr{T}_{n-2}^{ud}(\lambda)$. Let $a = 2(f-1) + \sum_{j=1}^{k-1} \mu_j + 1$. By Definition 2.14 and Lemma 2.1(d)-(e),

$$\begin{aligned} f_\mathfrak{t} E_{n-1} \equiv & E_{2f-1} T_{n,2f}^{-1} T_{n-1,2f-1}^{-1} \widetilde{\mathfrak{M}}_\lambda \sum_{j=a}^{n-1} q^{n-1-j} T_{n-1,j} F_\mathfrak{t} E_{n-1} \mod \mathscr{B}_n^{\rhd(f,\lambda)} \\ \equiv & E_{2f-1} E_{2f} \cdots E_{n-1} \widetilde{\mathfrak{M}}_\lambda \sum_{j=a}^{n-1} q^{n-1-j} T_{n-1,j} F_{\mathfrak{t},n} F_{\mathfrak{t},n-1} \\ & \times E_{n-1} \prod_{k=1}^{n-2} F_{\mathfrak{t},k} \mod \mathscr{B}_n^{\rhd(f,\lambda)}. \end{aligned}$$

By Lemma 2.1(b), $E_{n-1}\widetilde{\mathfrak{M}}_\lambda = \widetilde{\mathfrak{M}}_\lambda E_{n-1}$. Via Proposition 2.7, we write

$$E_{n-1} F_{\mathfrak{t},n-1} F_{\mathfrak{t},n} E_{n-1} = \Phi(L_1^\pm, \cdots, L_{n-2}^\pm) E_{n-1} \tag{4.7}$$

for some $\Phi(L_1^\pm, \cdots, L_{n-2}^\pm) \in F[L_1^\pm, L_2^\pm, \cdots, L_{n-2}^\pm] \cap \mathscr{B}_{n-2}$. On the other hand, by (2.10) , for any positive integer k, we have

$$E_{n-1} T_{n-2} L_{n-1}^k E_{n-1} = r L_{n-2}^k E_{n-1} + \omega \sum_{j=1}^{k} (L_{n-2}^{j-1} \omega_{n-1}^{(k-j+1)} - L_{n-2}^{2j-k-2}) E_{n-1}.$$

Acting σ to $E_{n-1} T_{n-2} L_{n-1}^k E_{n-1}$ yields the formula for $E_{n-1} T_{n-2}^{-1} L_{n-1}^{-k} E_{n-1}$. Using Lemma 2.1(c) for $i = n-2$ to rewrite $E_{n-1} T_{n-2}^{-1} L_{n-1}^{-k} E_{n-1}$ yields the formula for $E_{n-1} T_{n-2} L_{n-1}^{-k} E_{n-1}$. If $k = 0$, then $E_{n-1} T_{n-2} L_{n-1}^k E_{n-1} = r E_{n-1}$. So, there is a $\Psi(L_1^\pm, \cdots, L_{n-2}^\pm) \in F[L_1^\pm, L_2^\pm, \cdots, L_{n-2}^\pm] \cap \mathscr{B}_{n-2}$ such that

$$E_{n-1} T_{n-2} F_{\mathfrak{t},n} F_{\mathfrak{t},n-1} E_{n-1} = \Psi(L_1^\pm, \cdots, L_{n-2}^\pm) E_{n-1}. \tag{4.8}$$

By Definition 2.14, $E_{2f-1} E_{2f} \cdots E_{n-1} \widetilde{\mathfrak{M}}_\lambda = \mathfrak{M}_\mathfrak{u}$ where $\mathfrak{u} \overset{n-1}{\sim} \mathfrak{t}$ with $\mathfrak{u}_{n-1} = \lambda \cup \{(k+1, 1)\}$ if $\mu_k > 1$ and $\mathfrak{u}_{n-1} = \mathfrak{t}_{n-1} = \mu$ if $\mu_k = 1$. In the latter case, $\mathfrak{u} = \mathfrak{t}$.

Let Φ_λ (resp. Ψ_λ) be obtained from Φ (resp. Ψ) by using $c_{\mathfrak{t}^\lambda}^\pm(k)$ instead of $L_k^\pm$ in Φ (resp.Ψ). Note that $\widetilde{\mathfrak{M}}_\lambda T_j = q \widetilde{\mathfrak{M}}_\lambda$ for $a \le j \le n-3$. By Theorem 2.19 and the definition of $\mathfrak{u}$,

$$f_\mathfrak{t} E_{n-1} = (\Phi_\lambda + q[\mu_k - 1]\Psi_\lambda) \overline{\mathfrak{M}}_\mathfrak{u} + \sum_{\mathfrak{v} \overset{n-2}{\succ} \mathfrak{u}} b_\mathfrak{v} \overline{\mathfrak{M}}_\mathfrak{v} \prod_{k=1}^{n-2} F_{\mathfrak{t},k}.$$

We use Lemma 3.4(b) to express $\overline{\mathfrak{M}}_{\mathfrak{v}}$ as an F-linear combination of $f_{\mathfrak{s}}$'s. So, $\mathfrak{s} \succeq \mathfrak{v}$. Note that $L_i^{\pm}$ act on $f_{\mathfrak{s}}$ as scalars for all $1 \le i \le n$. So is $\prod_{k=1}^{n-2} F_{\mathfrak{t},k}$. Since we are assuming that $\mathfrak{v} \overset{n-2}{\succ} \mathfrak{u}$, $f_{\mathfrak{u}}$ can not appear in the expression of $\overline{\mathfrak{M}}_{\mathfrak{v}} \prod_{k=1}^{n-2} F_{\mathfrak{t},k}$. By Lemma 3.4(b), the coefficient of $f_{\mathfrak{u}}$ in $\overline{\mathfrak{M}}_{\mathfrak{u}}$ is 1. So, $E_{\mathfrak{t}\mathfrak{u}}(n-1) = \Phi_\lambda + q[\mu_k - 1]\Psi_\lambda$. We assume $\mathfrak{t} \neq \mathfrak{u}$. Then

$$\begin{aligned}
E_{\mathfrak{t}\mathfrak{u}}(n-1) f_{\mathfrak{t}} E_{n-1} =& (\Phi_\lambda + q[\mu_k - 1]\Psi_\lambda) f_{\mathfrak{t}} E_{n-1} \\
=& f_{\mathfrak{t}} E_{n-1}(1 + q[\mu_k - 1]T_{n-2}) F_{\mathfrak{t},n} F_{\mathfrak{t},n-1} E_{n-1} \qquad \text{by (4.7)-(4.8)} \\
=& \sum_{\mathfrak{v} \overset{n-1}{\sim} \mathfrak{t}} E_{\mathfrak{t}\mathfrak{v}}(n-1) f_{\mathfrak{v}} (1 + q[\mu_k - 1]T_{n-2}) F_{\mathfrak{t},n-1} F_{\mathfrak{t},n} E_{n-1} \\
=& E_{\mathfrak{t}\mathfrak{t}}(n-1) f_{\mathfrak{t}} E_{n-1} + q^2[\mu_k - 1] E_{\mathfrak{t}\mathfrak{t}}(n-1) f_{\mathfrak{t}} E_{n-1}
\end{aligned}$$

In the last equality, we use $f_{\mathfrak{v}} F_{\mathfrak{t},n} F_{\mathfrak{t},n-1} = 0$ (resp. $f_{\mathfrak{v}} T_{n-2} F_{\mathfrak{t},n} F_{\mathfrak{t},n-1} = 0$) for $\mathfrak{v} \overset{n-1}{\sim} \mathfrak{t}$ and $\mathfrak{v} \neq \mathfrak{t}$ which follows from Lemma 3.4(d) (resp. Lemma 3.4(d) and Lemma 3.5(a)). We also use Lemma 3.9(a) to get $f_{\mathfrak{t}} T_{n-2} = q f_{\mathfrak{t}}$. So, $E_{\mathfrak{t}\mathfrak{u}}(n-1) = [\mu_k] E_{\mathfrak{t}\mathfrak{t}}(n-1)$. We remark that the above equality holds true when $\mathfrak{u} = \mathfrak{t}$. One can verify it similarly. In this case, $\mu_k = 1$. Similar computation shows that $\Phi_\lambda = E_{\mathfrak{t}\mathfrak{t}}(n-1)$ and $\Psi_\lambda = q E_{\mathfrak{t}\mathfrak{t}}(n-1)$.

By similar arguments as above, we have

$$\begin{aligned}
& f_{\mathfrak{t}^\lambda \mathfrak{u}} f_{\mathfrak{u}\mathfrak{t}^\lambda} \\
\equiv& F_{\mathfrak{t}^\lambda} E_{2f-1} T_{n,2f}^{-1} T_{n-1,2f-1}^{-1} \widetilde{\mathfrak{M}}_\lambda F_{\mathfrak{u},n-1} F_{\mathfrak{u},n} \widetilde{\mathfrak{M}}_\lambda T_{2f-1,n-1}^{-1} T_{2f,n}^{-1} E_{2f-1} F_{\mathfrak{t}^\lambda} \mod \mathscr{B}_n^{\rhd(f,\lambda)} \\
\equiv& F_{\mathfrak{t}^\lambda} E_{2f-1} \cdots E_{n-2} \widetilde{\mathfrak{M}}_\lambda E_{n-1} F_{\mathfrak{u},n-1} F_{\mathfrak{u},n} E_{n-1} \widetilde{\mathfrak{M}}_\lambda E_{n-2} \cdots E_{2f-1} F_{\mathfrak{t}^\lambda} \mod \mathscr{B}_n^{\rhd(f,\lambda)} \\
\equiv& E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]! F_{\mathfrak{t}^\lambda} E_{2f-1} \cdots E_{n-1} \widetilde{\mathfrak{M}}_\lambda E_{n-2} \cdots E_{2f-1} F_{\mathfrak{t}^\lambda} \mod \mathscr{B}_n^{\rhd(f,\lambda)} \\
=& E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]! F_{\mathfrak{t}^\lambda} \overline{\mathfrak{M}}_\lambda T_{2f,n} T_{2f-1,n-1} E_{n-2} \cdots E_{2f-1} F_{\mathfrak{t}^\lambda} \\
=& E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]! F_{\mathfrak{t}^\lambda} \overline{\mathfrak{M}}_\lambda F_{\mathfrak{t}^\lambda} \\
\equiv& E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]! f_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} \mod \mathscr{B}_n^{\rhd(f,\lambda)}.
\end{aligned}$$

So, $\langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle = E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]!$. Since $\langle\ ,\ \rangle$ is associative, $\langle f_{\mathfrak{u}} E_{n-1}, f_{\mathfrak{t}} \rangle = \langle f_{\mathfrak{u}}, f_{\mathfrak{t}} E_{n-1} \rangle$. Thus $\langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle E_{\mathfrak{t}\mathfrak{u}}(n-1) = \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle E_{\mathfrak{u}\mathfrak{t}}(n-1)$. By Lemma 3.10(b),

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\delta^{f-1}[\mu]!} = \frac{1}{\delta^{f-1}[\mu]!} \frac{[\mu_k]^2 E_{\mathfrak{t}\mathfrak{t}}(n-1)}{E_{\mathfrak{u}\mathfrak{u}}(n-1)} E_{\mathfrak{u}\mathfrak{u}}(n-1)\delta^{f-1}[\lambda]! = [\mu_k] E_{\mathfrak{t}\mathfrak{t}}(n-1),$$

where $E_{\mathfrak{t}\mathfrak{t}}(n-1)$ can be computed explicitly by Lemma 3.10(a). $\square$

Proposition 4.9. *Suppose $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda) \in \Lambda_n$, and $l(\lambda) = l$. If $\hat{\mathfrak{t}} = \mathfrak{t}^\mu$, and $\mathfrak{t}_{n-1} = \mathfrak{t}_n \cup p$ with $p = (k, \mu_k)$ $k < l$, define $\mathfrak{u} = \mathfrak{t} s_{n,a+1}$ with $a = 2(f-1) + \sum_{j=1}^k \mu_j$ and $\mathfrak{v} = (\mathfrak{u}_1, \cdots, \mathfrak{u}_{a+1})$. Then*

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\delta^{f-1}[\mu]!} = \frac{[\mu_k] E_{\mathfrak{v}\mathfrak{v}}(a)}{r^2 q^{2(\mu_k - 2k)} - 1} \frac{\prod_{r_1 \in \mathscr{A}(\mu)^{<p}} (r^2 q^{-2(c_\mu(p) - c_\mu(r_1))} - 1)}{\prod_{r_2 \in \mathscr{R}(\mu)^{<p}} (r^2 q^{-2(c_\mu(p) + c_\mu(r_2))} - 1)}. \tag{4.10}$$

Proof. By definition, $\mathfrak{u} = \mathfrak{t}s_{n,a+1}$. Using the argument in the proof of Proposition 4.2, we have

$$\begin{aligned}\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle &= \langle f_{\mathfrak{u}}, f_{\mathfrak{u}}\rangle \prod_{j=a+2}^{n}\left(1-\omega^2\frac{c_{\mathfrak{u}}(j)c_{\mathfrak{u}}(a+1)}{(c_{\mathfrak{u}}(j)-c_{\mathfrak{u}}(a+1))^2}\right)\\ &= \frac{\langle f_{\mathfrak{u}}, f_{\mathfrak{u}}\rangle}{r^2q^{2(\mu_k-2k)}-1}\frac{\prod_{r_1\in\mathscr{A}(\mu)^{<p}}(r^2q^{-2(c_\mu(p)-c_\mu(r_1))}-1)}{\prod_{r_2\in\mathscr{R}(\mu)^{<p}}(r^2q^{-2(c_\mu(p)+c_\mu(r_2))}-1)}.\end{aligned}$$

By Proposition 4.1, $\langle f_{\mathfrak{u}}, f_{\mathfrak{u}}\rangle = \langle f_{\mathfrak{v}}, f_{\mathfrak{v}}\rangle \prod_{i=k+1}^{\ell}[\lambda_i]!$ where $\mathfrak{v} \in \mathscr{T}^{ud}_{a+1}(\nu)$ with $\mathfrak{v} = (\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_{a+1})$ and $\mathfrak{u}_{a+1} = \nu$. Finally, we use Proposition 4.5 and $[\mu_k]! = [\lambda_k]![\mu_k]$ to get $\langle f_{\mathfrak{v}}, f_{\mathfrak{v}}\rangle = E_{\mathfrak{v}\mathfrak{v}}(a)[\mu_k]^2\delta^{f-1}\prod_{i=1}^{k}[\lambda_i]!$. Simplifying $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle$ via previous formulae yields (4.10), as required. □

Definition 4.11. Suppose $(f,\lambda) \in \Lambda_n$ and $(\ell,\mu) \in \Lambda_{n-1}$. Write $(\ell,\mu) \to (f,\lambda)$ if either $\ell = f$ and $\mu = \lambda \setminus \{p\}$ or $\ell = f-1$ and $\mu = \lambda \cup \{p\}$. If $(\ell,\mu) \to (f,\lambda)$ we define $\gamma_{\lambda/\mu} \in F$ to be the scalar by declaring that

$$\gamma_{\lambda/\mu} = \frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle}{\delta^{\ell}[\mu]!} \tag{4.12}$$

where $\mathfrak{t} \in \mathscr{T}^{ud}_n(\lambda)$ with $\hat{\mathfrak{t}} = \mathfrak{t}^{\mu} \in \mathscr{T}^{ud}_{n-1}(\mu)$.

The following is the first main result of this paper.

Theorem 4.13. *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over $\mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$, where r, q are indeterminates and $\omega = q - q^{-1}$. The Gram determinant $\det G_{f,\lambda}$ associated to the cell module $\Delta(f,\lambda)$ of $\mathscr{B}_n$ can be computed by the following formula*

$$\det G_{f,\lambda} = \prod_{(\ell,\mu)\to(f,\lambda)} \det G_{\ell,\mu}\cdot \gamma_{\lambda/\mu}^{\dim\Delta(\ell,\mu)} \in \mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]. \tag{4.14}$$

Furthermore, each scalar $\gamma_{\lambda/\mu}$ can be computed explicitly by (4.3), (4.6), (4.10) and Lemma 3.10(a).

Proof. We first compute the Gram determinants over $\mathbb{C}(q^{\pm}, r^{\pm}, \omega^{-1})$. In order to use the results in section 4, we have to use the fundamental theorem of algebra (see [18]).

Since $\tilde{G}_{f,\lambda}$, defined via orthogonal basis of $\Delta(f,\lambda)$, is a diagonal matrix and each diagonal is of form $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle$, $\mathfrak{t} \in \mathscr{T}^{ud}_n(\lambda)$, we have $\det \tilde{G}_{f,\lambda} = \prod_{\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)}\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle$. By Proposition 4.1,

$$\det \tilde{G}_{f,\lambda} = \prod_{(\ell,\mu)\to(f,\lambda)} \det \tilde{G}_{\ell,\mu}\cdot \gamma_{\lambda/\mu}^{\dim\Delta(l,\mu)}.$$

However, by Lemma 3.4(f) $\det G_{f,\lambda} = \det \tilde{G}_{f,\lambda}$ and $\det G_{\ell,\mu} = \det \tilde{G}_{\ell,\mu}$. Since the Jucys-Murphy basis of $\Delta(f,\lambda)$ is defined over $\mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$, $\det G_{f,\lambda} \in \mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$. □

Recall that λ' is the dual partition of the partition λ. The following result gives the relation between the integral factors of $\det G_{f,\lambda}$ and $\det G_{f,\lambda'}$.

Corollary 4.15. *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over $\mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$, where $\omega = q - q^{-1}$. Suppose $(f, \lambda) \in \Lambda_n$ and $\varepsilon \in \{-1, 1\}$. Then $r - \varepsilon q^a$ is a factor of $\det G_{f,\lambda}$ if and only if $r + \varepsilon q^{-a}$ is a factor of $\det G_{f,\lambda'}$.*

Proof. Note that $p = (i, j)$ is an addable (resp. removable) node of λ if and only if $p' = (j, i)$ is an addable (resp. removable) node of λ'. Detailed analysis for the numerators and denominators of $\gamma_{\lambda/\mu}$ by elementary computation yields the result. □

If we consider q as a scalar, then $\prod_{(\ell,\mu)\to(f,\lambda)} \gamma_{\lambda/\mu}^{\dim \Delta(\ell,\mu)}$ can be considered as a rational function in r. Write $\prod_{(\ell,\mu)\to(f,\lambda)} \gamma_{\lambda/\mu}^{\dim \Delta(\ell,\mu)} = \frac{f(r)}{g(r)}$ such that the g.c.d of $f(r)$ and $g(r)$ is 1. The following result is useful when we determine the zero divisors of Gram determinants.

Corollary 4.16. *Let $f(r)$ and $g(r)$ be defined as above. Suppose $\varepsilon \in \{-1, 1\}$. If $r - \varepsilon q^a \mid f(r)$ and $r - \varepsilon q^a \nmid g(r)$ with $a \in \mathbb{Z}$, then $r - \varepsilon q^a$ is a factor of $\det G_{f,\lambda}$. In other words, $\det G_{f,\lambda} = 0$ if $r = \varepsilon q^a$.*

Proof. The result follows from the fact that $\det G_{f,\lambda} \in \mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$. □

5. Semisimplicity criteria for $\mathscr{B}_n$ over a field

In this section, we consider $\mathscr{B}_{n,F}$ over an arbitrary field F. We will give a necessary and sufficient condition for $\mathscr{B}_{n,F}$ being (split) semisimple. We will denote $\mathscr{B}_{n,F}$ by $\mathscr{B}_n$ if there is no confusion.

We remark that we may not have the orthogonal representations over F. However, we still have the recursive formula in (4.14) since the Gram matrix associated to each cell module is a matrix over R. In what follows, we will use this fact frequently.

Proposition 5.1. *Let $\mathscr{B}_n, n \geq 2$ be the Birman-Wenzl algebra over $\mathbb{Z}[r^{\pm}, q^{\pm}, \omega^{-1}]$. Then*

$$\begin{aligned} \det G_{1,(n-2)} = & q^{\frac{1}{2}(n-1)(3n-4)} \left(\frac{[n-2]!}{r(q^2-1)} \right)^{\frac{1}{2}n(n-1)} (r-q)^{\frac{1}{2}n(n-3)} \\ & \times (r+q^3)^{\frac{1}{2}(n-1)(n-2)} (r^2 - q^{6-2n})^{n-1} (r - q^{3-2n}). \end{aligned} \tag{5.2}$$

Proof. Let

- $(\mathfrak{s}_{k,2})_i = (i)$ for $i \leq k-1$ and $(\mathfrak{s}_{k,2})_i = (i-2)$ for $k \leq i \leq n$.
- Suppose $3 \leq j \leq k$. Define $(\mathfrak{s}_{k,j})_i = (i)$ for $i \leq j-2$ and $(\mathfrak{s}_{k,j})_i = (i-1, 1)$, for $j-1 \leq i \leq k-1$, and $(\mathfrak{s}_{k,j})_i = (i-2)$, $k \leq i \leq n$.

Then $\mathscr{T}_n^{ud}(\lambda) = \{\mathfrak{s}_{k,j} \mid 2 \leq j \leq k \leq n\}$. We use Proposition 4.1 and (4.3), (4.6), (4.10) to compute $\langle f_{\mathfrak{s}_{k,j}}, f_{\mathfrak{s}_{k,j}} \rangle$. We have

- $\langle f_{\mathfrak{s}_{2,2}}, f_{\mathfrak{s}_{2,2}} \rangle = \delta [n-2]!$,
- $\langle f_{\mathfrak{s}_{k,2}}, f_{\mathfrak{s}_{k,2}} \rangle = \frac{q^3 [n-2]!}{r(q^2-1)} \frac{r - q^{3-2k}}{r - q^{5-2k}} (r^2 q^{2k-6} - 1)[k-1]$ for $3 \leq k \leq n$,
- $\langle f_{\mathfrak{s}_{k,j}}, f_{\mathfrak{s}_{k,j}} \rangle = \frac{q}{r} \frac{[n-2]!}{q^2-1} (r-q)(r+q^3) \frac{[j-2]}{[j-1]} \frac{r^2 - q^{6-2k}}{r^2 - q^{8-2k}}$ for $3 \leq j \leq k \leq n$.

Thus

$$\prod_{k=3}^{n}\langle f_{\mathfrak{s}_{k,2}}, f_{\mathfrak{s}_{k,2}}\rangle = \left(\frac{q^3[n-2]!}{r(q^2-1)}\right)^{n-2}[n-1]!\frac{r-q^{3-2n}}{r-q^{-1}}\prod_{k=3}^{n}(r^2q^{2k-6}-1), \tag{5.3}$$

and

$$\begin{aligned}\prod_{3\le j\le k\le n}\langle f_{\mathfrak{s}_{k,j}}, f_{\mathfrak{s}_{k,j}}\rangle &= \left(\frac{q}{r}\frac{[n-2]!}{q^2-1}(r-q)(r+q^3)\right)^{\frac{(n-2)(n-1)}{2}} \\ &\quad\times\frac{(r^2-q^{6-2n})^{n-2}}{[n-1]!}\prod_{k=3}^{n}\frac{1}{r^2-q^{8-2k}}.\end{aligned}\tag{5.4}$$

Note that $\det G_{1,\lambda} = \prod_{2\le j\le k\le n}\langle f_{s_{k,j}}, f_{s_{k,j}}\rangle$. Now, (5.2) follows from elementary computation via equalities given above. □

Let F be a field containing non-zero $\mathbf{q}, \mathbf{r}$ and $(\mathbf{q}-\mathbf{q}^{-1})^{-1}$. The following results can be verified directly.

Lemma 5.5. *Suppose $o(\mathbf{q}^2) > n$ and $(f,\lambda)\in\Lambda_n$. For any $p_1\in\mathscr{A}(\lambda)$ and $p_2\in\mathscr{R}(\lambda)$, $[c_\lambda(p_1)+c_\lambda(p_2)]\neq 0$.*

Proposition 5.6. *Suppose that $n\ge 2$. Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over a field F. Suppose $o(\mathbf{q}^2) > n$ and $\mathbf{r}\notin\{\mathbf{q}^{-1}, -\mathbf{q}\}$. Then $\mathscr{B}_n$ is semisimple if and only if $\prod_{k=2}^{n}\det G_{1,(k-2)}\det G_{1,(1^{k-2})}\neq 0$.*

Proof. We use Lemma 5.5 to check that any factor $\mathbf{q}^a-\mathbf{q}^b$ in the numerators of (4.3), (4.6), (4.10) is not equal to zero if $o(\mathbf{q}^2) > n$. We will only consider the factors in $\det G_{f,\lambda}$ with forms $\mathbf{r}\pm\mathbf{q}^a$, $a\in\mathbb{Z}$. When we want to prove $\det G_{f,\lambda}=0$ for $\mathbf{r}=\varepsilon\mathbf{q}^a$, $\varepsilon\in\{1,-1\}$, we will consider $\mathbf{q}$ to be the indeterminate q first. In other words, we have $o(q)=\infty$. In order to get the result for $n<o(\mathbf{q}^2)<\infty$, $\mathbf{q}\in F$, we will specialize the indeterminate q to $\mathbf{q}\in F$. In the remainder of the proof, we consider $\mathbf{q}$ to be the indeterminate q. Also, we use r instead of $\mathbf{r}$.

($\Rightarrow$) Suppose $\det G_{1,(k-2)}=0$ for some $2\le k\le n$. We claim that there is a $(f,\lambda)\in\Lambda_n$ such that $\det G_{f,\lambda}=0$. It gives rise to a contradiction since the Gram determinant associated to any cell module is not equal to zero if a cellular algebra is semisimple.

If $\det G_{1,(1^{k-2})}=0$, then we use Corollary 4.15 and the above claim twice to find a partition λ such that $\det G_{f,\lambda}=0$. This will give rise to a contradiction, too. Consequently, $\prod_{2\le k\le n}\det G_{1,(k-2)}\det G_{1,(1^{k-2})}\neq 0$ if $\mathscr{B}_n$ is semisimple.

Now, we prove our claim. Since $\mathscr{B}_n$ is semisimple, $\det G_{1,(n-2)}\neq 0$. It is easy to verify $\det G_{1,\emptyset}\neq 0$ if $r\notin\{q^{-1},-q\}$. Therefore, we can assume $2<k<n$. By Proposition 5.1, $r\in\{q^{3-2k}, \pm q^{3-k}, -q^3, q\}$, $3<k<n$. The case $k=3$ has been dealt with in [7] which says $r\in\{q^{-3},\pm 1,-q^3\}$. One can use the program [19] (written in GAP language) to verify it directly. [4]

[4] F. Luebeck wrote the GAP program for Brauer algebras when he visited our department in 2006. Imitating his program, we wrote the GAP program for Birman-Wenzl algebras. We take this opportunity to express our gratitude to him.

Case 1. $n-k$ **is even**:
Let $f=\frac{n-k}{2}+1$ and $\lambda=(k-2)$. Then $(f,\lambda)\in\Lambda_n$. If $(f-1,\mu)\to(f,\lambda)$, then $\mu\in\{\mu_1,\mu_2\}$ and $\mu_1=(k-1)$, $\mu_2=(k-2,1)$. By Proposition 4.5, we have

$$r_{\lambda/\mu_1}r_{\lambda/\mu_2}=\frac{q^{2k-2}[k-2](r-q)(r+q^3)(r^2-q^{6-2k})^2(r-q^{3-2k})}{r^2(q^2-1)^2[k-1](r^2-q^{8-2k})(r-q^{5-2k})} \tag{5.7}$$

When $k>3$, $(r-q)(r+q^3)(r^2-q^{6-2k})(r-q^{3-2k})$ and $(r^2-q^{8-2k})(r-q^{5-2k})$ are co-prime[5] in $\mathbb{Z}[r^{\pm},q^{\pm},(q-q^{-1})^{-1}]$. If $k=3$, then $(r+q^3)(r^2-1)(r-q^{-3})$ and $(r+q)(r-q^{-1})$ are co-prime in $\mathbb{Z}[r^{\pm},q^{\pm},(q-q^{-1})^{-1}]$. Consequently, by (4.14), $\det G_{(f-1,\mu_1)}\det G_{(f-1,\mu_2)}\det G_{f,(k-3)}$ has to be divided by $(r^2-q^{8-2k})^{\dim\Delta(f-1,\mu_2)}(r-q^{5-2k})^{\dim\Delta(f-1,\mu_1)}$. Therefore, $\det G_{f,\lambda}=0$ if $\det G_{1,(k-2)}=0$.

Case 2. $n-k$ **is odd**:
There are three subcases we have to discuss.

Subcase 2a. $r=q$ **for some** $k>3$ **or** $r=-q^3$ **for some integer** k **with** $k>2$:
Let $f=\frac{n-k+1}{2}$ and $\lambda=(k-1)$. By (5.7) and Corollary 4.16, $\det G_{f,\lambda}=0$. We remark that we can use (5.7) since $\gamma_{\lambda/\mu}$ depends only on λ and μ.

Subcase 2b. $r=q^{3-2k}$**, for some integer** k **with** $2<k<n$:
Let $f=\frac{n-k+1}{2}$ and $\lambda=(k-2,1)$. Suppose $k>3$. If $(f-1,\mu)\to(f,\lambda)$, then $\mu\in\{\mu_1,\mu_2,\mu_3\}$ where $\mu_1=(k-1,1)$, $\mu_2=(k-2,2)$ and $\mu_3=(k-2,1,1)$. By Propositions 4.5, 4.9,

$$\begin{aligned}\gamma_{\lambda/\mu_1}&=\frac{q^3(r^2q^{2k-4}-1)(r^2q^{2k-8}-1)(r-q^{3-2k})}{r(q^2-1)(r-q^{5-2k})(r^2q^{2k-6}-1)},\\ \gamma_{\lambda/\mu_2}&=\frac{[k-3]q^2(rq-1)(r^2-q^{4-2k})(r^2-q^4)}{r[k-2](q^2-1)(r^2-q^{6-2k})(r-q)},\\ \gamma_{\lambda/\mu_3}&=\frac{q[k-1](r+q^5)(r^2-q^{8-2k})(r^2-q^4)}{r[k][2](q^2-1)(r^2-q^{10-2k})(r+q^3)}.\end{aligned}$$

If $k=3$, then $\lambda=(1,1)$. In this case, define $\mu_1=(2,1)$ and $\mu_2=(1,1,1)$. By Propositions 4.5, 4.9,

$$\gamma_{\lambda/\mu_1}=\frac{q^2(r^2-q^2)(rq+1)}{r(r^2-1)(q^2-1)}(r-q^{-3}),\ \text{and}\quad \gamma_{\lambda/\mu_2}=\frac{q(r+q^5)(r^2-q^2)}{r[3](q^2-1)(r+q^3)}.$$

Subcase 2c. $r=\pm q^{3-k}$**, for some integer** k **with** $2<k<n$:
If $k=3$, define $f=\frac{n}{2}$ and $\lambda=\varnothing$. If $(f-1,\mu)\to(f,\lambda)$, then $\mu=(1)$. If $(f-2,\nu)\to(f-1,\mu)$, then $\nu\in\{\nu_1,\nu_2\}$ where $\nu_1=(2)$ and $\nu_2=(1,1)$. By Propositions 4.5, 4.9,

$$\gamma_{\lambda/\mu}=\delta,\quad \gamma_{\mu/\nu_1}=\frac{q^3(r-q^{-3})(r^2-1)}{r(q^2-1)(r-q^{-1})},\quad \gamma_{\mu/\nu_2}=\frac{q(r+q^3)(r^2-1)}{[2]r(q^2-1)(r+q)}.$$

[5] since $o(q)=\infty$

Note that $r \notin \{q^{-1}, -q\}$. If $k = 4$, then $r = -q^{-1}$. Let $\lambda = (1,1,1)$. If $(f-1,\mu) \to (f,\lambda)$, then $\mu \in \{\mu_1, \mu_2\}$ where $\mu_1 = (2,1,1)$ and $\mu_2 = (1,1,1,1)$. By Propositions 4.5, 4.9,

$$r_{\lambda/\mu_1} = \frac{q^3(r-q^{-3})(r^2-q^4)(r+q^{-1})}{r(q^2-1)(r^2-q^2)}, \quad r_{\lambda/\mu_2} = \frac{q(r+q^7)(r^2-q^4)}{r[4](q^2-1)(r+q^5)}.$$

Suppose $k > 4$. Let $f = \frac{n-k+1}{2}$ and $\lambda = (k-3,1,1)$. If $(f-1,\mu) \to (f,\lambda)$, then $\mu \in \{\mu_1, \mu_2, \mu_3\}$ where $\mu_1 = (k-2,1,1)$, $\mu_2 = (k-3,2,1)$ and $\mu_3 = (k-3,1,1,1)$. By Propositions 4.5, 4.9,

$$\begin{aligned}
\gamma_{\lambda/\mu_1} &= \frac{q^{2k-5}(r-q^{5-2k})(r^2-q^{6-2k})(r^2-q^{12-2k})}{r(q^2-1)(r^2-q^{10-2k})(r-q^{7-2k})}, \\
\gamma_{\lambda/\mu_2} &= \frac{[k-4]q^3(r-q^{-1})(r^2-q^{6-2k})(r^2-q^6)(r+q)}{r[k-3](q^2-1)(r^2-q^{8-2k})(r^2-q^4)}, \\
\gamma_{\lambda/\mu_3} &= \frac{q[k-1](r+q^7)(r^2-q^{12-2k})(r^2-q^6)}{r[3][k](q^2-1)(r^2-q^{14-2k})(r+q^5)}.
\end{aligned}$$

In each case, by Corollary 4.16, $\det G_{f,\lambda}$ is divided by either $(r-q^{3-2k})(r \pm q^{3-k})(r+q^3)(r-q)$ for $k > 3$ or $(r-q^{-3})(r\pm 1)(r+q^3)$ for $k = 3$. Thus, $\det G_{f,\lambda} = 0$. This completes the proof of the claim.

($\Leftarrow$) Suppose that $\mathscr{B}_n$ is not semisimple. Then $\det G_{f,\lambda} = 0$ for some $(f,\lambda) \in \Lambda_n$. By (4.14), either $\det G_{\ell,\mu} = 0$ or the numerator of $\gamma_{\lambda/\mu}$ is equal to zero for some $(\ell,\mu) \to (f,\lambda)$. In the first case, by induction on n with $n \geq 3$, $\prod_{k=2}^{n-1} \det G_{1,(k-2)} \det G_{1,(1^{k-2})} = 0$, a contradiction. In the latter case, if $l = f$, then $\gamma_{\lambda/\mu} \neq 0$ since we are assuming that $o(q^2) > n$ and the numerator of $\gamma_{\lambda/\mu}$ is a product of $[k]$ for $k \leq n-1$. By Propositions 4.5, 4.9, we need only consider the numerators of $\gamma_{\lambda/\mu}$ with $\ell = f-1$. We claim that

a) $r^2 q^{c_\lambda(p)+c_\lambda(q)} \neq 1$ for all $p, q \in \mathscr{A}(\lambda)$.

b) $r^2 q^{c_\mu(q)-c_\mu(p)} \neq 1$ for all $q \in \mathscr{A}(\mu)$ and $p \in \mathscr{R}(\mu)$.

At first, we prove a). We assume that p (resp. q) is in the k-th (resp. ℓ-th) row. Since two boxes have the same contents if they are in the same diagonal, we can move both p and q to either the first row or the first column of a partition. Therefore, there is a partition $\xi \in \{(k-2), (1^{k-2})\}$ for some $2 \leq k \leq n$ such that $p_1, q_1 \in \mathscr{A}(\xi)$ with $c_\lambda(p) + c_\lambda(q) = c_\xi(p_1) + c_\xi(q_1)$. By our assumption and Proposition 5.2 and Corollary 4.15, $c_\lambda(p) + c_\lambda(q)$ is a factor of $\det G_{1,\xi}$. Since we are assuming that $\prod_{k=2}^{n} \det G_{1,(k-2)} \det G_{1,(1^{k-2})} \neq 0$, we have $\det G_{1,\xi} \neq 0$, forcing $c_\lambda(p) + c_\lambda(q) \neq 0$.

b) can be proved by similar arguments as above. We leave the details to the reader. By Propositions 4.5 and 4.9, and our claim b), $\gamma_{\lambda/\mu} = 0$ implies either $E_{\mathfrak{t}\mathfrak{t}}(n-1) = 0$ or $E_{\mathfrak{v}\mathfrak{v}}(a) = 0$, where $\mathfrak{t}, \mathfrak{v}$ and a are defined in (4.12) and Proposition 4.9, respectively.

Using our claims a)-b), we have that $rq^{2c_\lambda(p_1)} - q^{-1} = 0$ or $rq^{2c_\lambda(p_1)} + q = 0$ if $E_{\mathfrak{t}\mathfrak{t}}(n-1)E_{\mathfrak{v}\mathfrak{v}}(a) = 0$. In this case, μ is obtained from λ by adding the addable node p_1.

If $c_\lambda(p_1) = 0$. then $r \in \{q^{-1}, -q\}$, a contradiction. So, we can assume that $c_\lambda(p_1) \neq 0$. First, we deal with the case when $c_\lambda(p_1) > 0$. Note that $r \in \{q^{-(1+2c_\lambda(p_1))}, -q^{1-2c_\lambda(p_1)}\}$. In the first case, by (5.2), $\det G_{1,\eta} = 0$ where $\eta = (c_\lambda(p_1))$. Note that $c_\lambda(p_1) \leq n-2$, by assumption, $\det G_{1,\eta} \neq 0$, a contradiction. Assume that $r = -q^{1-2c_\lambda(p_1)}$. By Proposition 5.2, $\det G_{1,\eta} = 0$ where $\eta = (2c_\lambda(p_1))$. Since we are assuming that $\prod_{m=2}^n \det G_{1,(m-2)} \det G_{1,(1^{m-2})} \neq 0$, we have $n-2 < 2c_\lambda(p_1) = 2(\lambda_k + 1 - k) \leq 2\lambda_k$, forcing $k = 1$. By (5.7), the numerators of $\gamma_{\lambda/\mu_1}\gamma_{\lambda/\mu_2}$ must be divided by $r + q^{1-2c_\lambda(p_1)}$, where μ_1, μ_2 is the same as those in (5.7) and the k in (5.7) should be replaced by $\lambda_1 + 2$ which is equal to $n - 2f + 2$. Thus, $r + q^{1-2c_\lambda(p_1)} \in S$ where

$$S = \{r - q, r + q^3, r \pm q^{1-\lambda_1}, r - q^{-(1+2\lambda_1)}\}.$$

On the other hand, we have $\lambda_1 + 2 \leq n$ since we are assuming that $f \geq 1$. By (5.2), $\prod_{m=2}^n \det G_{1,(m-2)} \det G_{1,(1^{m-2})}$ is divided by each element in S. This implies that $r + q^{1-2c_\lambda(p_1)} \neq 0$, a contradiction.

If $c_\lambda(p_1) < 0$, we use Corollary 4.15 to consider $r \in \{-q^{1+2c_\lambda(p_1)}, q^{2c_\lambda(p_1)-1}\}$. In this situation, we still get a contradiction by the result for $c_\lambda(p_1) > 0$ stated above. □

Proposition 5.8. *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over a field F containing the parameters $\mathbf{q}^\pm, \mathbf{r}^\pm$ and $(\mathbf{q} - \mathbf{q}^{-1})^{-1}$. Assume $\mathbf{r} \in \{\mathbf{q}^{-1}, -\mathbf{q}\}$.*

- *a) $\mathscr{B}_n$ is not semisimple if n is either even or odd with $n \geq 7$.*
- *b) $\mathscr{B}_1$ is always semisimple.*
- *c) $\mathscr{B}_3$ is semisimple if and only if $o(\mathbf{q}^2) > 3$ and $\mathbf{q}^4 + 1 \neq 0$.*
- *d) $\mathscr{B}_5$ is semisimple if and only if $o(\mathbf{q}^2) > 5$ and $\mathbf{q}^6 + 1 \neq 0$, and $\mathbf{q}^8 + 1 \neq 0$, and* $\operatorname{char} F \neq 2$.

Proof. We use r, q instead of $\mathbf{r}, \mathbf{q}$ in the proof of this result. Since $r \in \{q^{-1}, -q\}$, $\delta = 0$. Suppose that n is even. Let $a = \dim_F \Delta(n/2 - 1, (1))$. By (4.14), $\det G_{n/2,\varnothing} = \det G_{n/2-1,(1)}\delta^a = 0$. We have $\det G_{1,(3,2)} = 0$ when $r \in \{q^{-1}, -q\}$. One can use [19] to verify the above formulae easily. This shows that $\mathscr{B}_7$ is not semisimple. We also use [19] to get the following formulae:

- $\det G_{1,(1)} = (q^4 + 1)$ if $r \in \{q^{-1}, -q\}$.
- $\det G_{1,(3)} = 2^5[2]^{10}[3]^{14}(1 + q^8)$ (resp. $-[2]^{10}[3]^{11}q^{-2}(1 + q^4)^6$) if $r = -q$ (resp. if $r = q^{-1}$).
- $\det G_{1,(1,1,1)} = q^{-2}[3](1 + q^4)^6$ (resp. $2^5[3]^4(1 + q^8)$) if $r = -q$ (resp. $r = q^{-1}$).
- $\det G_{1,(2,1)} = -q^2[2]^4[3]^{15}(1 + q^6)^4$ if $r \in \{q^{-1}, -q\}$.
- $\det G_{2,(1)} = -32q^2(1 + q^2)(1 + q^4)^{10}(1 + q^6)$ if $r \in \{q^{-1}, -q\}$.

Now, (b)-(d) follow from the results on the semisimplicity of Hecke algebras $\mathscr{H}_n$ for $n \in \{1, 3, 5\}$ together with the above formulae.

We close the proof by showing that $\det G_{\frac{n-5}{2},(3,2)} = 0$ for all odd n with $n > 7$. This can be verified by comparing the recursive formulae on $\det G_{\frac{n-5}{2},(3,2)}$ with $\det G_{1,(3,2)}$. We leave the details to the reader. □

Theorem 5.9. *Let $\mathscr{B}_n$ be the Birman-Wenzl algebra over a field F which contains non-zero parameters r, q, ω, where $\omega = q - q^{-1}$.*

a) Suppose $r \notin \{q^{-1}, -q\}$.

- (a1) *If $n \geq 3$, then $\mathscr{B}_n$ is semisimple if and only if $o(q^2) > n$ and $r \notin \cup_{k=3}^{n} \{q^{3-2k}, \pm q^{3-k}, -q^{2k-3}, \pm q^{k-3}\}$.*
- (a2) *$\mathscr{B}_2$ is semisimple if and only if $o(q^2) > 2$.*
- (a3) *$\mathscr{B}_1$ is always semisimple.*

b) Assume $r \in \{q^{-1}, -q\}$.

- (b1) *$\mathscr{B}_n$ is not semisimple if n is either even or odd with $n \geq 7$.*
- (b2) *$\mathscr{B}_1$ is always semisimple.*
- (b3) *$\mathscr{B}_3$ is semisimple if and only if $o(q^2) > 3$ and $q^4 + 1 \neq 0$.*
- (b4) *$\mathscr{B}_5$ is semisimple if and only if $o(q^2) > 5$, $q^6 + 1 \neq 0$, and $q^8 + 1 \neq 0$ and $\operatorname{char} F \neq 2$.*

Proof. Suppose $n \neq 2$. Theorem 5.9 follows from Propositions 5.6, (5.2) and Corollary 4.15, (resp. Proposition 5.8) under the assumption $r \notin \{q^{-1}, -q\}$ (resp. $r \in \{q^{-1}, -q\}$). When $n = 2$, we compute $\det G_{1,\varnothing}$ directly to verify the result. $\square$

Let $\delta = \frac{(q+r)(qr-1)}{r(q+1)(q-1)}$. Then

$$\operatorname{limit}_{q \to 1} \delta \in \{1, 2, \cdots, n-2\} \cup \{-2, -4, \cdots, 4-2n\} \cup \{-1, -2, \cdots, 4-n\}$$

if $r \in \cup_{k=3}^{n} \{q^{3-2k}, \pm q^{3-k}, -q^{2k-3}, \pm q^{k-3}\}$ and $n \geq 3$. They are the parameters we got in [17] such that the corresponding Brauer algebra is not semisimple. Finally, we remark that some partial results on Brauer algebras being semisimple over $\mathbb{C}$ can be found in [5, 6, 20].

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